Interpretable Sparse Proximate Factors for Large Dimensions

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Abstract

This paper approximates latent statistical factors with sparse and easy-to-interpret proximate factors. Latent factors in a large-dimensional factor model can be estimated by principal component analysis, but are usually hard to interpret. By shrinking factor weights, we obtain proximate factors that are easier to interpret. We show that proximate factors consisting of 5-10% of the cross-section observations with the largest absolute loadings are usually sufficient to almost perfectly replicate the population factors, without assuming a sparse structure in loadings. We derive analytical asymptotic bounds for the correlation of appropriately rotated proximate factors with the population factors based on extreme value theory. These bounds provide guidance on how to construct the proximate factors. Simulations and empirical applications to financial portfolio and macroeconomic data illustrate that proximate factors approximate latent factors well while being interpretable.

Keywords: Factor Analysis, Principal Components, Sparse Loading, Interpretability, Large-Dimensional Panel Data, Large N and T

JEL classification: C14, C38, C55, G12
1 Introduction

“Big data” is becoming increasingly popular to study problems in economics and finance. Large-dimensional datasets contain rich information and open the possibilities of new findings and new efficacious models, which is feasible only if we better understand the datasets. Factor modeling (Bai and Ng, 2002; Bai, 2003; Fan et al., 2013) is a method that summarizes information in large-dimensional panel data and is an active area of research. In a large-dimensional factor model, both the time dimension and cross-section dimension of the datasets are large and most of the co-movement can be explained by a few factors. Latent factors estimated from the data are particularly appealing as the underlying factor structure is usually not known. These factors are usually estimated by Principal Component Analysis (PCA). Latent PCA factors have been successfully used in economics and finance, for example for prediction and forecasting (Stock and Watson, 2002a,b), asset pricing of many assets (Lettau and Pelger, 2018b; Kelly et al., 2018), high-frequency asset return modeling (Ait-Sahalia and Xiu, 2018; Pelger, 2019), conditional risk-return and term structure analysis (Ludvigson and Ng, 2007, 2009) and optimal portfolio construction (Fan et al., 2013).1

However, as the latent PCA factors are linear combinations of all cross-sectional units, they are usually hard to interpret. This poses a challenge for modeling and understanding the underlying structure to explain the co-movement in the data. This becomes particularly relevant for the applications in economics and finance where the objective is to understand the underlying economic mechanism.

Practitioners and academics alike have used an intuitive approach to interpret latent statistical factors by focusing on the largest factor weights. A pattern in the largest factor weights suggests an economic interpretation (e.g. Lettau and Pelger (2018b); Pelger (2019)). In this paper we formalize this idea and show that the factors that are only based on the largest factor weights, provide already an excellent approximation to the population factors. This step further reduces the dimensionality of the problem helping to better understand the economic mechanism in the problem.

We propose easy-to-interpret proximate factors for latent factors. We exploit the insight that cross-section units with larger factor weights have a larger signal-to-noise ratio, hence providing more information about the underlying factors. Our method consists of four simple steps. First, we estimate the underlying factor structure with the conventional Principal Component Analysis (PCA), which returns the weights to construct the latent PCA factors. Second, we set all factor weights to zero except for the largest absolute ones. Third, the

1Of course, the application of latent factor models goes beyond the applications listed above, e.g. inferring missing values in matrices (Candès and Tao, 2010; Candès et al., 2011).
proximate factors are obtained from a simple regression on the thresholded factor weights. Finally, the loadings are obtained from a regression on the proximate factors. We work under a general scenario where the factor weights and loadings in the true model are not sparse.

We show that one needs to make a clear distinction between factor weights and loadings. In a conventional PCA analysis the loadings serve two purposes: First, they measure the exposure of the cross-sectional data to the factors; Second, they correspond to weights to construct the latent PCA factors. The same view is for example taken in a sparse PCA setup where loadings and hence also the factor weights are shrinked to a sparse matrix. We show that in an approximate factor model with non-sparse factor weights and loadings, we can construct proximate factors with sparse factor weights that are very close to the population factors. These proximate factors have non-sparse loadings which are consistent estimates of the true population loadings. The proximate factors are considerably easier to interpret as they are only based on a small fraction of the data, while enjoying the same properties as the non-sparse factors.²

We develop the statistical arguments that explain why the sparse proximate factors can be used as substitutes for the non-sparse PCA factors. The conventional derivations used in large-dimensional factor modeling to prove consistency of estimated factors do not apply to our proximate factors. It turns out that the proximate factors are in general a biased estimate of the true population factors. We can control this bias with Extreme Value Theory (EVT) and show how to construct the proximate factors such that this bias becomes negligible. The closeness between proximate factors and true factors is measured by the generalized correlation.³ We provide an asymptotic probabilistic lower bound for the generalized correlation based on EVT. The lower bound is easy to calculate and depends on the tail distribution of population factor weights and the number of nonzero elements in the sparse factor weights. The lower bound provides guidance on constructing the proximate factors that are guaranteed to have a high correlation with the true factors. Moreover, when factor weights have unbounded support, we show that proximate factors asymptotically span the same space as latent factors. Importantly, the estimated loadings of the proxy factors con-

²A common method to interpret low-dimensional factor models is to find a rotation of the common factors with a meaningful interpretation. This approach uses the insight that factor models are only identified up to an invertible transformation and represent the same model after an appropriate rotation. The varimax criterion proposed by Kaiser (1958) is a popular way to select factors whose factor weights have groups of large and negligible coefficients. However, in large-dimensional factor models with non-sparse factor weight structure, finding a “good” rotation becomes considerably more challenging. It is generally easier with our sparse proximate factors to find a rotation that has a meaningful interpretation.

³Generalized correlation (also called canonical correlation) measures how close two vector spaces are. It has been studied by Anderson (1958) and applied in large-dimensional factor models (Bai and Ng, 2006; Pelger, 2019; Andreou et al., 2017; Pelger and Xiong, 2018).

⁴In simulations, we verify that the lower bound has good finite sample properties.
verge to the true population loadings up to the usual rotation. This surprising result is due to the two stage procedure for estimating the loadings that averages out the idiosyncratic noise. Hence, regressions based on the more interpretable proxy factors will asymptotically yield the same results as using the harder to interpret PCA factors.

Our results are of practical and theoretical importance. First, we provide a simple and easy-to-implement method to approximate latent factors by a small number of observations. These sparse proximate factors usually provide a much simpler economic interpretation of the model, either by directly analyzing the sparse composition or after rotating them appropriately. Second, we show in empirical and simulation studies that this approximation works surprisingly well and almost no information is lost by working with the sparse factors. Third, our asymptotic bounds for the correlations between the proximate factors and the population factors provides the theoretical reasoning why our method works so well. In particular, it clarifies under which assumptions the proximate factors are a good approximation or even converge to the population factors. As mentioned before, the idea of analyzing the largest factor weights is not new, but we are the first to provide the theoretical arguments why and when it is a reasonable approach. Fourth, the asymptotic bounds provide guidance on how to select the key tuning parameter for our estimator, i.e. the number of non-zero elements.

We need to overcome three major challenges, when showing the probabilistic lower bound for the generalized correlation. First, we need to show that the estimated loadings converge uniformly to some rotation of the true loadings, under the general approximate factor model assumptions. Uniform consistency of the estimated loadings is necessary for our argument that cross-section units with the largest estimated loadings have the largest probabilistic “signal” for the underlying factors. Second, the hard-thresholding procedure that sets most loadings to zero to get the sparse factor weights has the down-side that we lose some of the large sample properties in the cross-section dimension. We need to take into account the cross-section dependency structure in the errors among a few cross-section units, which directly affects the noise level in calculating the generalized correlation. Third, the sparse factor weights, as well as proximate factors, are in general not orthogonal to one another in contrast to their non-sparse estimated version. A proximate factor for a specific population factor can also be correlated with other latent factors, which is reflected in the generalized correlation.

5Obviously, the degree of sparsity can be chosen to obtain a sufficiently high generalized correlation with the estimated PCA factors. However, this would be an in-sample choice of the tuning parameter with no guarantee for its out-of-sample performance. Our theoretical bound provides an alternative to select the tuning parameter based on arguments that should also hold out-of-sample.

6We impose assumptions similar to Bai and Ng (2002). In contrast to Fan et al. (2013), we do not assume loadings to be fixed and bounded which would restrict the distributions from which the loadings can be sampled.
correlation. Our assumptions and results need to take this complex inter-factor correlations into account.

In two empirical applications, we apply our method to a large number of financial characteristic-sorted portfolios and a large-dimensional macroeconomic dataset. We find that in both datasets, proximate factors with around 5-10% of the cross-sectional units can very well approximate the non-sparse PCA factors with average correlations of around 97.5%. The proximate factors explain almost the same amount of variation as the non-sparse PCA factors. The sparse factors have an economic meaningful interpretation which would be hard to obtain from the non-sparse representation.

Sparse loadings have already been employed to reduce the composition of factors and to make latent factors more interpretable. Most work formulates the estimation of sparse loadings as a regularized optimization problem to estimate principal components with a soft thresholding term, such as an \( \ell_1 \) penalty term or an elastic net penalty term (Zou et al., 2006; Mairal et al., 2010; Bai and Ng, 2008; Bai and Liao, 2016). An alternative approach is to take a Bayesian perspective and specify sparse priors for factor loadings and use Bayesian updating to obtain posteriors for sparse loadings (Lucas et al., 2006; Bhattacharya and Dunson, 2011; Kaufmann and Schumacher, 2013; Pati et al., 2014). All these approaches assume that loadings are sparse in the population model, which allows these approaches to develop an asymptotic inferential theory. Nevertheless, the assumption of sparse population loadings may not be satisfied in many datasets. For example, the exposure to a market factor is universal and non-sparse in equity data. It is important to understand that this line of work typically does not distinguish between factor weights and loadings, i.e. it uses the shrinked loadings for factors weights and exposure. In contrast we only estimate sparse factor weights, but non-sparse loadings. Additionally, we do not assume that the population factor weights are sparse. Of course, it is possible to adjust the sparse PCA method to make this important distinction between factor weights and loadings, i.e. loadings are obtained from a second stage regression. However, in contrast to our approach there is no statistical theory explaining and justifying this approach in the same general setup that we are using. Furthermore, lasso-type estimators have the well-known shortcoming that they create biased estimates by the way how they are shrinking large elements and a similar non-optimal shrinking happens in our case. It turns out that even the modified sparse PCA performs worse than our method in simulations and empirical applications.

Another method to increase the understandability and interpretability of factors is to associate latent factors or factor loadings with observed variables. Some latent factors can

\[ \text{(Choi et al., 2010; Lan et al., 2014; Kawano et al., 2015)} \] estimate the sparse loadings by minimizing the sum of the negative log-likelihood of the data with a soft thresholding term.
be approximated well by observed economic factors, such as Fama-French factors for equity data (Fama and French, 1992) or level, slope, and curvature factors for bond data (Diebold and Li, 2006). Fan et al. (2016a) propose robust factor models to exploit the explanatory power of observed proxies on latent factors. Another approach is to model how the factor loadings relate to observable variables. Connor and Linton (2007), Connor et al. (2012), and Fan et al. (2016b) at least partially employ subject-specific covariates to explain factor loadings, such as market capitalization, price-earning ratios, and other firm characteristics. However, in order to explain latent factors by observed variables, it is necessary to include all the relevant variables, some of which might not be known. Our sparse proximate factors can provide discipline on which assets and covariates to focus on.

The rest of the paper is structured as follows. Section 2 introduces the model and the estimator. Section 3 shows the consistency result for the estimated loadings and the asymptotic probabilistic lower bound for the generalized correlation. Section 4 presents simulation results. In section 5 we apply our approach to financial and macroeconomic datasets. Section 6 concludes the paper. All proofs and additional empirical results are delegated to the Appendix.

2 Model Setup

2.1 Estimator

We assume that a large dimensional panel data set \( X \in \mathbb{R}^{N \times T} \) with \( T \) time-series and \( N \) cross-sectional observations has a factor structure. There are \( K \) common factors in \( X \) and both \( T \) and \( N \) are large:

\[
X = \Lambda F^\top + e, \tag{1}
\]

where \( F \in \mathbb{R}^{T \times K} \), \( \Lambda \in \mathbb{R}^{N \times K} \), and \( e \in \mathbb{R}^{N \times T} \) are common factors, factor loadings, and idiosyncratic components. Factors and loadings are unobserved. A \( K \)-factor model from the panel data can be estimated by Principal Component Analysis (PCA). The PCA factors

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8Our proximate factors also provide a justification for the construction of Fama-French style factors. Lettau and Pelger (2018b) among others show that PCA-type factors explain well double-sorted portfolio data. The largest loadings (factor weights) for these PCA factors are exactly the extreme quantiles and our proximate factors essentially coincide with the long-short Fama-French-type factors.

9We assume that we have consistently estimated the number of factors \( K \).

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\( \hat{F} \in \mathbb{R}^{T \times K} \) and loadings \( \hat{\Lambda} \in \mathbb{R}^{N \times K} \) minimize a quadratic loss function

\[
\{\hat{F}, \hat{\Lambda}\} = \arg \min_{F, \Lambda} \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} (X_{it} - \lambda_i^T f_t)^2,
\]

(2)

where \( X_{it} \) is the observation of the \( i \)-th cross-section unit at time \( t \), \( f_t = [f_{t,1}, \cdots, f_{t,K}]^T \in \mathbb{R}^{K \times 1} \) is the factor value at time \( t \), and \( \lambda_i = [\lambda_{i,1}, \cdots, \lambda_{i,K}]^T \in \mathbb{R}^{K \times 1} \) is the factor loading of the \( i \)-th cross-section unit. Factors and loadings are identified up to an invertible transformation, as for any invertible \( H, H\hat{\lambda}_i \) and \( (H^T)^{-1} \hat{f}_t \) also minimize the objective function (2).

Under the standard identification assumption that \( \frac{1}{N}\hat{\Lambda}^T \hat{\Lambda} \) is an identity matrix and \( \frac{1}{T} \hat{F}^T \hat{F} \) is a diagonal matrix, the solution to equation (2) is

\[
\frac{1}{NT} XX^\top \hat{\Lambda} = \hat{\Lambda} \hat{S}
\]

(3)

\[
\hat{F} = \frac{1}{N} X^\top \hat{\Lambda}
\]

(4)

where \( \hat{\Lambda} \) are the eigenvectors of the \( K \)-largest eigenvalues of \( \frac{1}{NT} XX^\top \) multiplied by \( \sqrt{N} \). \( \hat{S} = \text{diag}(\hat{s}_1, \hat{s}_2, \cdots, \hat{s}_K) \) is a diagonal matrix with the \( K \) largest eigenvalues in descending order, and \( \hat{F} \) are the coefficients from regressing \( X \) on \( \hat{\Lambda} \). Note, that \( \hat{\Lambda} \) serves two purposes here: One the one hand it is the cross-sectional weight to construct the estimated factors and other hand it is the estimated exposure to the factors. \( \text{Bai (2003)} \) shows that in an approximate factor model \( \hat{\Lambda} \) and \( \hat{F} \) are consistent estimators of \( \Lambda \) and \( F \) up to an invertible transformation.

The general approximate factor model framework assumes that the loadings \( \Lambda \) and thus their consistent estimator \( \hat{\Lambda} \) are not sparse, which is a necessary assumption to allow for cross-sectional correlation in the idiosyncratic component. As the estimated factors are linear combinations of the cross-section units \( X \) weighted by a non-sparse \( \hat{\Lambda} \), they are composed of almost all cross-section units, which are hard to interpret.

We propose a method to estimate proximate factors that are sparse and hence more interpretable. The method is based on the following steps:

1. **Sparse factor weights**: \( \hat{\Lambda} \) are the standard PCA estimates of the loadings. The proximate factor weights \( \hat{\tilde{W}} \) are the largest elements of \( \hat{\Lambda} \) obtained as follows: We shrink the \( N - m \) smallest entries in absolute values in each estimated loading vector \( \hat{\lambda}_k \) to 0 and only keep the largest \( m \) elements to get the sparse weight vector \( \hat{\tilde{W}}_k \) for each factor \( k \). We standardize each sparse weight vector to have length one. \(^{10}\)

\(^{10}\)Formally, denote \( M = [M_1, M_2, \cdots, M_K] \in \mathbb{R}^{N \times K} \) as a mask matrix indicating which factor weights are
2. **Proximate factors**: We regress $X$ on $\tilde{W}$ to obtain the proximate factors $\tilde{F}$:

$$\tilde{F} = X^\top \tilde{W}(\tilde{W}^\top \tilde{W})^{-1}. \quad (6)$$

3. **Loadings of proximate factors**: We regress $X$ on $\tilde{F}$ to obtain the loadings $\Lambda$ of the proximate factors:

$$\Lambda = X^\top \tilde{F}(\tilde{F}^\top \tilde{F})^{-1}. \quad (7)$$

Proximate factors $\tilde{F}$ approximate latent factors $F$ well as measured by the generalized correlation. The generalized correlation equals the correlation between appropriately rotated proximate and population factors as defined in the next section. It is natural to measure the distance between two factors by their correlation. If two factors are perfectly correlated they explain the same variation in the data and provide the same results in linear regressions.\(^{11}\)

We illustrate the intuition in a one-factor model. In this case, the generalized correlation is equal to the squared correlation between $\tilde{F}$ and $F$. For simplicity, assume that the factors and idiosyncratic component are i.i.d. over time:

$$f_{t,1} \overset{iid}{\sim} (0, \sigma_f^2), \quad e_{1t} \overset{iid}{\sim} (0, \sigma_e^2), \quad f_1^\top f_1/T \rightarrow \sigma_f^2, \quad e_1^\top e_1/T \rightarrow \sigma_e^2.$$

Furthermore, our proximate factor consists of only one cross-sectional observation, i.e. $m = 1$. Without loss of generality, the nonzero entry in $\tilde{W}_1 \in \mathbb{R}^{N \times 1}$ is $\tilde{w}_{1,1}$ which is normalized set to zero. $M_j$ has $m$ 1s and $N - m$ 0s. The sparse factor weights $\tilde{W}$ can be written as

$$\tilde{W} = \left[ \frac{\hat{\Lambda}_1 \odot M_1}{\|\hat{\Lambda}_1 \odot M_1\|} \quad \frac{\hat{\Lambda}_2 \odot M_2}{\|\hat{\Lambda}_2 \odot M_2\|} \quad \ldots \quad \frac{\hat{\Lambda}_K \odot M_K}{\|\hat{\Lambda}_K \odot M_K\|} \right], \quad (5)$$

where $\hat{\Lambda}_j$ is $j$-th estimated loading. The vector $M_j$ has the element 1 at the position of the $m$ largest loadings of $\hat{\Lambda}_j$ and zero otherwise. $\odot$ denotes the Hadamard product for element by element multiplication of matrices.

\(^{11}\)Obviously perfectly correlated factors do not necessarily need to have the same mean. However, in our empirical study the first and second moment properties of the proximate factor coincide with the non-sparse PCA factors.
to \( \tilde{w}_{1,1} = 1 \). The squared correlation between \( \tilde{F} \) and \( F \) equals

\[
corr(\tilde{F}, F)^2 = \frac{(F^\top \tilde{F}/T)^2}{(F^\top F/T)(\tilde{F}^\top F/T)} = \left( \frac{f_1^\top (F_1\lambda_{1,1} + e_1)/T}{(f_1^\top F_1/T)^{1/2}((F_1\lambda_{1,1} + e_1)^T(F_1\lambda_{1,1} + e_1)/T)^{1/2}} \right)^2
\]

\[
\to \frac{\lambda_{1,1}^2}{\lambda_{1,1}^2 + 1/s^2},
\]

where \( s = \sigma_f/\sigma_e \) denotes the signal-to-noise ratio. The second equation follows from \( \tilde{F} = X^\top \tilde{W}(\tilde{W}^\top \tilde{W})^{-1} = X^\top \tilde{W} = (F_1\lambda_{1,1} + e_1)\tilde{w}_{1,1} = (F_1\lambda_{1,1} + e_1) \). The correlation increases with the size of \( |\lambda_{1,1}| \) and the signal-to-noise ratio \( s \). If the largest population loading is sufficiently large, the correlation will be close to one. In the rest of the paper, we formalize this idea under a general setup.

### 2.2 Assumptions

We impose several assumptions that are close to, but slightly stronger than, those in Bai and Ng (2002). In order to show \( \tilde{F} \) is “close” to \( F \) measured by the generalized correlation, \( \hat{\Lambda} \) needs to be a uniform consistent estimator for \( \Lambda \) up to some invertible transformation. Furthermore, the largest elements in \( |\Lambda_j| \in \mathbb{R}^{N \times 1} \) have to almost coincide with the largest elements in \( |\hat{\Lambda}_j| \), which requires uniform consistency. The following assumptions are necessary for all theorems in this paper. We assume that there exists a positive constant \( M < \infty \) that can be used in all the assumptions.

**Assumption 1.** Factors:

\[
E\|f_t\|^4 \leq M < \infty \text{ and } \frac{1}{T} \sum_{t=1}^{T} f_t f_t^\top \xrightarrow{P} \Sigma_F \text{ for some } K \times K \text{ positive definite matrix } \Sigma_F.
\]

**Assumption 2.** Factor loadings:

\[
E\|\lambda_i\|^4 \leq M < \infty \text{ and } \frac{1}{N} \sum_{i=1}^{N} \lambda_i \lambda_i^\top \xrightarrow{P} \Sigma_\Lambda \text{ for some } K \times K \text{ positive definite matrix } \Sigma_\Lambda. \text{ Loadings are independent of factors and errors.}
\]

**Assumption 3.** Time and Cross-Section Dependence and Heteroskedasticity: Denote \( E[e_i e_j] = \tau_{ij,ts}, \sigma_e^2 = \max_{i,j,t,s} |\tau_{ij,ts}|. \) Then for all \( N \) and \( T \),

1. \( E[e_{it}] = 0, E|e_{it}|^8 \leq M; \)
2. \( e_t \) is stationary, \( \Sigma_e = [\sigma_{e_{ij}}]_{N \times N} \) is the covariance matrix of \( e_t, \|\Sigma_e\|_1 \leq M; \)
3. \( \forall i, |\tau_{iti,s}| \leq |\tau_{ts}| \) for some \( \tau_{ts}, \frac{1}{T} \sum_{s=1}^{T} \sum_{t=1}^{T} |\tau_{ts}| \leq M; \)
4. \( E[T^{-1/2}(e_i^\top e_j - E e_i^\top e_j)]^4 < M. \)
Assumption 4. For all \( i \leq N, t \leq T, \) \( E \left| T^{-1/2} \sum_{t=1}^{T} f_t e_{it} \right|^4 < M. \)

These are standard assumptions for the general approximate factor model. Since \( \Sigma_e \) in Fan et al. (2013). This assumption restricts the cross-section dependence of errors and is from the stationarity of \( t \) in Bai and Ng (2002). This assumption implies that \( \sigma^2 \) hold. Assumption 3.1 imposes moment conditions for errors, which is the same as Assumption C.1 in Bai and Ng (2002) and Bai (2003) show that factors and loadings can be estimated consistently with PCA under Assumptions 1-4 when \( N,T \) to tently with PCA under Assumptions 1-4 when \( N,T \rightarrow \infty. \) More precisely, there exists an invertible \( H, \) such that \( \frac{1}{T} \sum_{t=1}^{T} \left\| \hat{f}_t - H^{-1} f_t \right\|^2 = O_p(1/N + 1/T) \) and \( \frac{1}{N} \sum_{i=1}^{N} \left\| \hat{\lambda}_i - H \Lambda_i \right\|^2 = O_p(1/N + 1/T). \) However, in general, consistency does not hold for proximate factors \( \tilde{F} \) for the following reason:

\[
\tilde{F} = X^\top \tilde{W} (\tilde{W}^\top \tilde{W})^{-1} = FA^\top \tilde{W} (\tilde{W}^\top \tilde{W})^{-1} + e^\top \tilde{W} (\tilde{W}^\top \tilde{W})^{-1}
\]

and \( e^\top \tilde{W}_k = \sum_{i=1}^{N} e_{t,i} \tilde{w}_{i,k} = \sum_{i=1}^{m} e_{t,j_i,k} \tilde{w}_{j_i,k,k} \) where \( j_i,k \) is the nonzero element in \( \tilde{W}_k, \) i.e. the sums are only taken over the non-zero weight entries. Even in the special case when the idiosyncratic components at time \( t \) are i.i.d with variance \( \sigma^2, \) the variance of the \( (t,k) \)-th entry in \( e^\top \tilde{W} \) is \( Var(\sum_{i=1}^{m} e_{t,j_i,k} \tilde{w}_{j_i,k,k}) = \sigma^2, \) which does not vanish as \( N,T \rightarrow \infty. \)

The average of the idiosyncratic component over the non-sparse loadings leads to a law of

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12 Assumption 4 about the population factors is the same as Bai and Ng (2002). Assumption 2 allows loadings to be random. Since the loadings are independent of factors and errors, all results in Bai and Ng (2002) hold. Assumption 3.1 imposes moment conditions for errors, which is the same as Assumption C.1 in Bai and Ng (2002). This assumption implies that \( \sigma^2 \) is bounded. Assumption 3.2 is close to Assumption 3.2 (i) and (ii) in Fan et al. (2013). This assumption restricts the cross-section dependence of errors and is standard in the literature on approximate factor models. Since \( \Sigma_e \) is symmetric, \( \| \Sigma_e \|_1 \leq M \) is equivalent to \( \| \Sigma_e \|_\infty \leq M. \) \( \| \Sigma_e \|_1 \leq M \) implies that \( \sum_{j=1}^{N} |r_{ij,t,t} | \leq M. \) Together with \( r_{ij,t,t} \) being the same for all \( t \) from the stationarity of \( e_t, \) this assumption implies Assumption C.3 in Bai and Ng (2002). Assumption 3.3 allows for weak time-series dependence of errors, which is slightly stronger than Assumption C.2 in Bai and Ng (2002). Assumption 3.6 is the time average counterpart of Assumption C.5 in Bai and Ng (2002). Assumption 4 implies Assumption D in Bai and Ng (2002). The fourth moment conditions in Assumptions 3.6 and 5, together with Boole’s inequality or the union bound, are used to show the uniform convergence of loadings, without assuming the boundedness of loadings. Since Assumptions 4 imply Assumption A-D in Bai and Ng (2002), all results in Bai and Ng (2002) hold.
large number that diversifies away the idiosyncratic components. For proximate factors this average is taken over a finite number of observations, resulting in the loss of diversification.

Although proximate factors are in general inconsistent, they can still be very close to the population factors. We will study the correlation respectively generalized correlation between the population and proximate factors. Our results depend on the following theorem that shows the uniform consistency of the estimated loadings.

**Theorem 1.** Under Assumptions 1-4,\
\[
\max_{1 \leq t \leq T} \left\| \hat{\lambda}_t - H \lambda_t \right\| = O_p(\sqrt{1/T} + N^{1/4}/\sqrt{T}).
\]

Theorem 1 states that the maximum difference between the estimated loading and some rotation of the true loading for any cross-section unit converges to 0 at a specific rate. Compared to \[
\frac{1}{N} \sum_{t=1}^{N} \left\| \hat{\lambda}_t - H \lambda_t \right\|^2 = O_p(1/N + 1/T),
\]
the uniform convergence rate of \[
\left\| \hat{\lambda}_t - H \lambda_t \right\|
\]
is slower if \( T/N^{3/2} = o(1) \). Theorem 1 states that a large \(|\hat{\lambda}_{t,j}|\) implies a large \(|\lambda_{t,j}|\). As a result, the \( m \)-th largest values of \(|\hat{\lambda}_j|\) is close to the \( m \)-th largest \(|(\Lambda H)_{-,j}|\), which is formally stated in Lemma 3 and proven in the Appendix. Hence, we can derive the distribution of the correlation \( corr(\tilde{F}, F) \) based on the largest population instead of estimated loadings.

### 3.2 Loadings of Proximate Factors

The estimated loadings of the proximate factors asymptotically span the same space as the population loadings and hence will lead to the same regression or projection results. Hence, up to an invertible transformation the loadings of the proximate factors are consistent. This result actually does not depend on the degree of sparsity \( m \). The key element is that the loadings are obtained in a second stage regression that diversifies away idiosyncratic noise. Hence, it is important to distinguish between loadings and factor weights as sparse loadings are in general not consistent estimator of the population loadings.

One of the major problems when comparing two different sets of loadings is that a factor model is only identified up to invertible linear transformations. Two sets of loadings represent the same model if the loadings span the same vector space. As proposed by Bai and Ng (2006)
the generalized correlation is a natural candidate measure to describe how close two vector spaces are to each other. Intuitively, we calculate the correlation between the loadings of the proximate and the population factors after rotating them appropriately. The generalized correlation measures of how many loading vectors two sets have in common. The generalized correlation between the loadings of proximate and population factors is defined as

$$\rho_{\tilde{\Lambda},\Lambda} = \text{tr} \left( (\Lambda^\top \Lambda/N)^{-1}(\Lambda^\top \tilde{\Lambda}/N)(\tilde{\Lambda}^\top \tilde{\Lambda}/N)^{-1}(\tilde{\Lambda}^\top \Lambda/N) \right).$$

Here, the generalized correlation $\rho_{\tilde{\Lambda},\Lambda}$, ranging from 0 to $K$ (number of factors), measures how close $\tilde{\Lambda}$ and $\Lambda$ are. If $\tilde{\Lambda}$ lies in the space spanned by $\Lambda$, then $\rho = K$. Otherwise, if the space spanned by $\tilde{\Lambda}$ is orthogonal to the space spanned by $\Lambda$, then $\rho = 0$.

We need to impose one additional weak assumption on the residuals:

**Assumption 5.** The largest eigenvalue of $\frac{1}{N} \mathbf{e}^\top \mathbf{e}$ is $o_p(1)$.

Assumption 5 is very weak and essentially satisfied in any sensible approximate factor model. It is standard and has been imposed in many related papers, e.g. Fan et al. (2013). It is slightly stronger than Assumption 3.3, that implies that the largest eigenvalue of the population residual auto-covariance matrix is bounded. Under suitable additional assumptions on the tail behavior of the residuals, Assumption 3.3 implies that $\| \frac{1}{N} \mathbf{e}^\top \mathbf{e} \|_2$ is $O_p(1)$ which would be sufficient. We denote by $W$ the matrix of the $m$ largest factor weights based on the population loadings, i.e. it follows the same definition as $\tilde{W}$ but applied to the population loadings $\Lambda$.

**Theorem 2.** Under Assumptions 1-4 and 5 and if $\| (W^\top \Lambda)^{-1} \|_2 = O_p(1)$, then it holds that

$$\rho_{\tilde{\Lambda},\Lambda} \overset{p}{\to} K. \quad (8)$$

The assumption that $\| (W^\top \Lambda)^{-1} \|_2 = O_p(1)$ is essentially a full rank assumption on $W^\top \Lambda$. It requires that the sparse set of cross-section units, that we use to construct the proximate factors, is affected by all factors in a non-redundant way. In the case of only one factor, i.e. $K = 1$, it is trivially satisfied. In the case of multiple factors, we need to rule out that the largest elements of two loading vectors are identical. This is an assumption on

---

15Our generalized correlation measure is the sum of the squared individual generalized correlations. The first individual generalized correlation is the highest correlation that can be achieved through a linear combination of the proximate loadings $\tilde{\Lambda}$ and the populations loadings $\Lambda$. For the second generalized correlation we first project out the subspace that spans the linear combination for the first generalized correlation and then determine the highest possible correlation that can be achieved through linear combinations of the remaining $K - 1$ dimensional subspaces. This procedure continues until we have calculated the $K$ individual generalized correlations. Mathematically the individual generalized correlations are the square root of the eigenvalues of the matrix $(\Lambda^\top \Lambda/N)^{-1}(\Lambda^\top \tilde{\Lambda}/N)(\tilde{\Lambda}^\top \tilde{\Lambda}/N)^{-1}(\tilde{\Lambda}^\top \Lambda/N)$. 

---
the tail dependency of the loadings. If for example the loading vectors are independent and \( m \geq K \) then this condition is satisfied.

Our notation of consistency does not imply point-wise consistency of the loadings, i.e. for a finite number of loading elements \( \tilde{\Lambda} \) can be different from \( \Lambda H \) where \( H \) is an invertible \( K \times K \) matrix. Our notation of consistency measures the asymptotic difference between vectors whose length goes to infinity and hence a finite number of elements have a negligible effect. The generalized correlation measure is the appropriate measure if we intend to use loadings for projections or in a cross-sectional regression. The strong result in Theorem 2 states that cross-sectional regressions with loadings of proxy factors yield the same results as using the population loadings up to an invertible matrix \( H \). Note, that this theorem has broader implications that go beyond proximate factors. For example it justifies why an iterative procedure to estimate latent factors leads to a consistent estimator after a few iterations.\(^\text{16}\)

Next, we will show that the proximate factors themselves are also very close to the population factors.

### 3.3 One-Factor Case

We start with the one-factor model and derive two characterizations for the correlation between the population and the proximate factor. The first characterization is based on a counting statistic, while the second one uses extreme value theory. In both cases, we derive analytical solutions for the lower bound. We use the results for comparative statics and preparing for the more general case.

If the population loadings \( \lambda_{i,1} \) are i.i.d., we have a closed-form lower bound for the asymptotic exceedance probability of the squared correlation \( \rho = \frac{(\tilde{F}^\top \tilde{F})^2}{F^\top FF^\top F} \).

**Proposition 1.** Assume Assumptions \([4]\) hold and that for all \( i \), the true loadings \( \lambda_{i,1} \) are i.i.d. with \( F_{|\lambda_{i,1}}(y) = P(|\lambda_{i,1}| \leq y) \) being continuous for all \( y \). Then for a given threshold \( 0 < \rho_0 < 1 \) and a fixed \( m \), we have

\[
\lim_{N,T \to \infty} P(\rho > \rho_0) \geq 1 - \lim_{N \to \infty} \sum_{j=0}^{m-1} \binom{N}{j} (1 - F_{|\lambda_{i,1}}(y_m))^j F_{|\lambda_{i,1}}(y_m)^{N-j},
\]

where \( y_m = \sqrt{\frac{1+h(m) \sigma^2}{m} \frac{\rho_0}{\sigma^2 + 1-\rho^2}} \).

---

\(^{16}\)Instead of applying PCA, latent factors can also be estimated by an iterative procedure where for a set of candidate factors a first stage set of loadings is estimated with a time-series regression, which is then used in a second stage to obtain factors in a cross-sectional regression. This procedure is iterated until convergence. For example Bai and Ng (2017) use a variation of this approach.
Proposition 2. In Proposition 1, if \( F|_{\lambda_{i,1}}(y_m) < 1 \), then as \( N, T \to \infty \),

\[
P(\rho > \rho_0) \to 1.
\]

Proposition 2 states that if the loadings have unbounded support the correlation between the proximate and population factor converges to one. At a first glance it seems to be at odds with our previous observations that the idiosyncratic component in a proximate factor cannot be diversified away. The intuition behind this consistency result is based on the growing signal-to-noise ratio. If loadings are sampled with an unbounded support independently of the idiosyncratic component, then for growing \( N \) the largest loadings are unbounded and their signal-to-noise ratio explodes. Hence with high probability the largest loadings do not coincide with large idiosyncratic movements and the variation of these cross-section units is essentially only explained by the factor. Hence, selecting the cross-section unit with the largest loading is close to picking the factor itself.

Note that after proper rescaling loadings with unbounded support can also be interpreted as approximately sparse population loadings. Hence, a sparse estimator is consistent if the true population model is sparse itself. The important contribution of this paper is that we can also characterize the asymptotic properties of the sparse estimator if the population model is not sparse.

Proposition 1 allows us to derive comparative statics. Denote the right-hand side of inequality (9) as \( p \) and \( x = F|_{\lambda_{i,1}}(y_m) \). We take the partial derivate of \( \bar{p} \) with respect to \( x \),

\[
\frac{\partial}{\partial x} \sum_{j=0}^{m-1} \binom{N}{j} (1-x)^j x^{N-j} = Nx^{N-1} + \sum_{j=1}^{m-1} \binom{N}{j} (1-x)^{j-1} x^{N-j-1} (N-j-Nx)
\]

\[
= m \binom{N}{m} x^{N-m} (1-x)^{m-1} > 0.
\]

\[
\frac{\partial}{\partial x} \sum_{j=0}^{m-1} \binom{N}{j} (1-x)^j x^{N-j} \text{ is increasing in } x, \text{ so } \bar{p} \text{ is decreasing in } F|_{\lambda_{i,1}}(y_m). \text{ If } F|_{\lambda_{i,1}}(y_m) = 1, \text{ it implies that the support of } |\lambda_{i,1}| \text{ is smaller than the threshold } y_m \text{ with probability 1, therefore } \bar{p} = 0. \text{ If } F|_{\lambda_{i,1}}(y_m) < 1, \text{ it implies that as } N \to \infty, |\Lambda_1| \text{ will have at least } m \text{ entries greater than } y_m, \text{ as stated in Proposition 2.} \text{ Thus, we have:}
\]

1. The larger \( \rho_0 \), the larger \( y_m \) and the smaller the exceedance probability \( \bar{p} \);  
2. The larger the signal-to-noise ratio \( \sigma_f/\sigma_e \), the smaller \( y_m \) and the larger \( \bar{p} \);  
3. The more dispersed the distribution of \( |\lambda_{1,i}| \), the larger \( \bar{p} \);  
4. The larger the cross-section dependence of errors \( h(m) \), the smaller \( \bar{p} \).
5. The number of nonzero elements \( m \) affects \( \rho \) in two ways: First, in most cases, \( \frac{1 + h(m)}{m} \)
decreases with \( m \), which raises \( \rho \). Second, larger \( m \) results in more subtraction terms
in \( \rho \) with the opposite effect leading to a trade-off.

Another perspective providing a lower bound for \( \rho \) employs EVT, which relaxes the i.i.d.
assumption of \( \lambda_{i,1} \). The lower bound depends only on the distribution of the extreme values
of loadings which can be modeled by extreme value theory under general conditions (e.g. 
\cite{Leadbetter1989, Hsing1988}).

We denote the largest \( |\lambda_{i,1}| \) by \( |\lambda_{(1),1}| \). Under the assumptions of EVT there exist sequences
of constants \( \{a_{1,N} > 0\} \), \( \{b_{1,N}\} \) such that \( P(|\lambda_{(1),1} - b_{1,N}| / a_{1,N} \leq z) \rightarrow G_1^*(z) \),
where \( G_1^*(z) = \exp \left\{ - \left[ 1 + \xi (\frac{z - \mu^*}{\sigma^*}) \right]^{-1/\xi} \right\} \), \( \mu^* = \mu - \frac{\sigma}{\xi} (1 - \theta^{-\xi}) \), \( \sigma^* = \sigma \theta^\xi \). \( \mu, \sigma, \xi \) are
parameters of the GEV distribution to characterize the tail distribution of i.i.d. random variables
with the same marginal distributions as \( |\lambda_{i,1}| \) and \( \theta \in (0, 1] \) is an extremal index measuring
the auto-dependence of \( |\lambda_{i,1}| \) in the tails. The results for the \( m \)-th-largest loading (extreme
order statistic) in the case of dependent loadings is more complex. In Lemma 1 in the Appendix
we provide the limiting distribution for the extreme order statistic of a strictly stationary
sequence \( |\lambda_{i,1}| \) satisfying the strong mixing condition\(^{17} \). Lemma 1 is adapted from Theorem
3.3 in \cite{Hsing1988}. It provides the necessary and sufficient condition such that there
exists a sequence \( u_{1,N}(\tau) \) and function \( G_{1,m}(\tau) \) with \( \lim_{N \rightarrow \infty} P(|\lambda_{(m),1}| \leq u_{1,N}(\tau)) = G_{1,m}(\tau) \).

**Theorem 3.** Suppose Assumptions \( \mathcal{A} \) holds and in addition the assumptions in Lemma
1 are satisfied s.t. \( \lim_{N \rightarrow \infty} P(|\lambda_{(m),1}| \leq u_{1,N}(\tau)) = G_{1,m}(\tau) \) for some sequence \( u_{1,N}(\tau) \) and
function \( G_{1,m}(\tau) \). For a given \( m \) we have

\[
\lim_{N,T \rightarrow \infty} P \left( \rho > \frac{m \sigma_{1,N}^2 u_{1,N}^2(\tau)}{(1 + h(m)) \sigma_{\xi}^2 + m \sigma_{1,N}^2 u_{1,N}^2(\tau)} \right) \geq 1 - G_{1,m}(\tau). \tag{10}
\]

**Theorem 3** provides a threshold such that the asymptotic probability of \( \rho \) exceeding this
threshold is larger than \( 1 - G_{1,m}(\tau) \). This probability lower bound is characterized by the
tail distribution of \( |\lambda_{i,1}| \) and the dependence structure in \( |\lambda_{i,1}| \). A special case is \( m = 1 \) for
which \( G_{1,1}(\tau) = e^{-\tau} \). If the tail distribution of \( |\lambda_{i,1}| \) follows an extreme value distribution
with parameters \( \mu, \sigma, \xi \) and a nonzero extremal index \( \theta^{18} \) then there exist sequences \( a_{1,N} \)
and \( b_{1,N} \) such that \( u_{1,N}(\tau) = a_{1,N} \left( \mu^* + \sigma^* \left( \frac{\tau - 1}{\xi} \right) \right) + b_{1,N} \) with \( \mu^* = \mu - \frac{\sigma}{\xi} (1 - \theta^{-\xi}) \) and

\(^{17}|\lambda_{i,1}| \) is indexed by the cross-section units. We assume that \( |\lambda_{i,1}| \) are exchangeable and can be properly
reshuffled to satisfy the strong mixing condition.

\(^{18}\)The extremal index \( \theta \) can be interpreted as the reciprocal of limiting mean cluster size. \cite{Smith1994, Weissman1978, Ancona2000} and others have studied methods to estimate \( \theta \). A
common method is the blocks method, which divides the data with \( n \) observations into approximate \( k_n \)
blocks of length \( r_n \), where \( n \approx k_n r_n \). The estimated \( \hat{\theta}_n = Z_n / N_n \) is the ratio of number of blocks in which
there is at least one exceedance to the total number of exceedances in all observations.
Thus, given a threshold \( \rho \), we have \( \lim_{N \to \infty} P \left( |\lambda_{(m),1}| \leq a_{1,N} \left( \mu^* + \sigma^* \left( \frac{\tau - \xi}{\xi} \right) \right) + b_{1,N} \right) = e^{-\tau} \) or equivalently \( \lim_{N \to \infty} P \left( (|\lambda_{(m),1}| - b_{1,N})/a_{1,N} \leq z \right) = \exp \left\{ - \left[ 1 + \xi \left( \frac{z-\mu}{\sigma} \right) \right]^{-1/\xi} \right\}$. 

Another special case is \( |\lambda_{i,1}| \) being independent, which yields \( G_{1,m}(\tau) = \lim_{N \to \infty} P(|\lambda_{(m),1}| \leq u_{1,N}(\tau)) = e^{-\tau} \sum_{l=0}^{m-1} \frac{t^l}{l!} \). We immediately obtain the following corollary for independent loadings:

**Corollary 1.** Assume Assumptions 1-4 holds, the loadings are i.i.d. and the largest absolute loading element \( |\lambda_{(1),1}| \) follows an extreme value distribution \( P((|\lambda_{(1),1}| - b_{1,N})/a_{1,N} \leq z) \to G_{1,1}(z) \) for some sequences of constants \( \{a_{1,N} > 0\} \) and \( \{b_{1,N}\} \) and \( G_{1,1}(z) = e^{-\tau(z)} \) with \( \tau(z) = \left[ 1 + \xi \left( \frac{z-\mu}{\sigma} \right) \right]^{-1/\xi} \). Then, the \( m \)-th largest absolute loading element \( |\lambda_{(m),1}| \) satisfies \( G_{1,m}(\tau) = \lim_{N \to \infty} P\left( (|\lambda_{(m),1}| - b_{1,N})/a_{1,N} \leq z \right) = e^{-\tau(z)} \sum_{s=0}^{m-1} \frac{\tau(z)^s}{s!} \). For fixed \( m \) it holds

\[
\lim_{N \to \infty} P \left( \rho > \frac{ma_1^2(a_{1,N}z + b_{1,N})^2}{(1+h(m))\sigma^2 + ma_1^2(a_{1,N}z + b_{1,N})^2} \right) \geq 1 - G_{1,m}(\tau).
\] (11)

The sequences \( a_{1,N} \) and \( b_{1,N} \) determine to which of the Gumbel, Frechet and Weibull families the tail distribution of \( |\lambda_{i,1}| \) belongs. Here are some examples:

1. **Gumbel distribution:**
   
   (a) Exponential distribution (\( \lambda_{1,i} \sim exp(1) \)): \( a_{1,N} = 1, b_{1,N} = N \).
   (b) Standard normal distribution: \( a_{1,N} = \frac{1}{N\phi(b_{1,N})} \) and \( b_{1,N} = \Phi^{-1}(1 - 1/N) \), where \( \phi(\cdot), \Phi(\cdot) \) are the pdf and cdf of a standard normal variable.

2. **Frechet distribution** (\( F(x) = exp(-1/x) \)): \( a_{1,N} = N, b_{1,N} = 0 \).

3. **Weibull distribution** (\( \lambda_{1,i} \sim Uniform(0,1) \)): \( a_{1,N} = 1/N, b_{1,N} = 1 \).

We denote the threshold in Corollary 1 by \( \rho_0 = \frac{ma_1^2(a_{1,N}z + b_{1,N})^2}{(1+h(m))\sigma^2 + ma_1^2(a_{1,N}z + b_{1,N})^2} \). Given \( z \) and \( \rho_0 \), the more dependent the \( \lambda_{i,1} \), the smaller the extremal index \( \theta \) and the smaller \( 1 - G_{1,1}(z) \). Thus, given a threshold \( \rho_0 \), the probabilistic lower bound decreases with the dependence level of \( |\lambda_{i,1}| \). Moreover, \( \rho_0 \) increases with \( \sigma_{f_1} \) and decreases with \( \sigma_e \), which implies \( \rho \) tends to be larger with a larger signal-to-noise ratio \( \sigma_{f_1}/\sigma_e \). It is straightforward to verify that \( \rho_0 \) is non-decreasing with \( N \) if \( a_{1,N} \) and \( b_{1,N} \) are among the previously listed examples of extreme value distributions. These findings are aligned with those implied by Proposition 1.

\(^{19}\text{See Theorem 5.2 in Coles et al. (2001)}\)
3.4 Multi-Factor Case

The arguments of the one-factor model extend to a model with multiple factors. As our simulations and empirical results illustrate, the simple thresholding method provides proximate factors that explain very well the non-sparse PCA factors. However, formalizing the properties of the lower bound are more challenging. First, we have to work with the generalized correlation instead of simple correlation between the proximate and population factors. Second, the sparse factor weight vectors are in general not orthogonal to each other in contrast to the PCA-loadings. In order to derive sharp theoretical bounds, we will impose additional assumptions. However, these assumptions are only necessary for deriving analytical asymptotic results, but not for our estimator to work as verified with simulated and empirical data.

One of the major problems when comparing two different sets of factors is that a factor model is only identified up to invertible linear transformations. We will again use the generalized correlation to measure how many factors two sets have in common. The generalized correlation between proximate factors and population factors is defined as

\[ \rho = tr \left( (F^\top F/T)^{-1} (F^\top \tilde{F}/T)(\tilde{F}^\top \tilde{F}/T)^{-1}(\tilde{F}^\top F/T) \right) . \]

Here, the generalized correlation \( \rho \), ranging from 0 to \( K \) (number of factors), measures how close \( F \) and \( \tilde{F} \) are. If \( \tilde{F} \) lies in the space spanned by \( F \), then \( \rho = K \).\(^{20}\)

We study two cases: First, the sparse weight vectors are orthogonal to each other, which allows us to directly extend the one-factor results to the multi-factor case; Second, we first find an appropriate rotation of the estimated loadings before thresholding them. We only assume that the sparse rotated factor weights are orthogonal, which is weaker. In our empirical examples we observe that several proximate factors are composed of the same small number of cross-section units but with different weights. In this case it is possible to find a rotation of the factors such that the proximate factors are composed of a disjoint set of cross-sectional units.

For the first case we assume that the sparse factor weights are “non-overlapping”. Formally, we define “non-overlapping” as\(^ {21}\)

\[ \sum_{i=1}^{N} \mathbb{1} \left( \sum_{j=1}^{K} \mathbb{1}(\tilde{w}_{i,j} \neq 0) > 1 \right) = 0, \]

\(^{20}\)The individual generalized correlations are the square root of the eigenvalues of the matrix \((F^\top F/T)^{-1}(F^\top \tilde{F}/T)(\tilde{F}^\top \tilde{F}/T)^{-1}(\tilde{F}^\top F/T)\).

\(^{21}\)\( \mathbb{1}(.) \) denotes an indicator function and is one if the condition is satisfied and zero otherwise.
which means that at most one factor weight is nonzero for every cross-section unit in the sparse factor weights. Then, the results from the one-factor model directly generalize to the multi-factor case. In this case the sparse factor weights are orthonormal, i.e. \( \tilde{W}^\top \tilde{W} = I_K \), similar to the non-sparse \( \hat{\Lambda} \). The generalized correlation \( \rho \) equals the sum of \( K \) squared correlations between each proximate factor and rotated true factor.

We need to impose additional assumptions on the population model to obtain the non-overlapping sparse factor weights.

**Assumption 6.** For the rotated population loadings \( V = \Lambda H \in \mathbb{R}^{N \times K} \) and a given finite \( m \), we denote the \( m \)-dimensional vector of elements in \( V_j \in \mathbb{R}^N \) with \( m \) largest absolute values as \( \tilde{V}_j \in \mathbb{R}^m \). We assume that the cumulative density function of \( v_{i, \cdot} \in \mathbb{R}^K \) is continuous and that \( \tilde{V}_j \) and \( \tilde{V}_k \) are asymptotically independent for \( j \neq k = 1, \ldots, K \). Furthermore, for each loading vector \( |V_j| \), the entry with the \( m \)-th largest absolute value, \( |v_{(m), j}| \), satisfies the assumptions in Lemma[1] yielding

\[
G_{j,m}(\tau) = \lim_{N \to \infty} P(|v_{(m), j}| \leq u_{j,N}(\tau)) = e^{-\tau} \left[ 1 + \sum_{l=1}^{m-1} \frac{\tau^l}{l!} \left( \sum_{i=l}^{m-1} \pi_{j}^{(i)}(i) \right) \right], \quad (12)
\]

where \( \pi_{j}^{(i)}(i) \) is defined analogously to (20) in Lemma[1].

Under Assumption 6 the largest rotated population loadings are asymptotically “non-overlapping.” Then Theorem[1] implies that also the sparse factor weights are “non-overlapping” with high probability. Furthermore, Assumption 6 implies a joint distribution for the largest elements in \( V \):

\[
P(|v_{(m),1}| \leq u_{1,N}(\tau_1), \ldots, |v_{(m),K}| \leq u_{K,N}(\tau_K)) \to \prod_{j=1}^{K} G_{j,m}(\tau_j)
\]
as the extreme values for different columns of \( V \) are independently distributed.

An additional complication in the multi-factor case is the relationship between the sparse and non-sparse eigenvectors. We show that an asymptotic lower bound for \( \rho \) is \( K - (1 + h(m))\sigma^2 tr \left( \left( \frac{1}{K} \tilde{W}^\top (\Lambda F^\top F \Lambda^\top) \tilde{W} \right)^{-1} \right) \). The complication in the multi-factor case arises from the fact that \( \tilde{W}^\top \Lambda \) is in general not a diagonal matrix. In order to illustrate this point we will consider the simple example of \( m = 1 \) and \( K = 2 \), i.e. a two factor model where the proximate factors only take the largest loading values. W.l.o.g. we assume the first element \( \lambda_{1,1} \) is the largest element for the first vector \( \Lambda_1 \) and the second element \( \lambda_{2,2} \) is the largest
element for $\Lambda_2$. Then, we have

$$
\tilde{W}^\top \Lambda = \begin{pmatrix}
\tilde{w}_{1,1} & 0 \\
0 & \tilde{w}_{2,2} \\
0 & 0 \\
\vdots & \vdots 
\end{pmatrix}
\begin{pmatrix}
\lambda_{1,1} & \lambda_{2,1}
\lambda_{1,2} & \lambda_{2,2}
\lambda_{1,3} & \lambda_{2,3}
\vdots & \vdots
\end{pmatrix}
\approx
\begin{pmatrix}
\tilde{w}_{1,1} \lambda_{1,1} & \tilde{w}_{1,1} \lambda_{2,1}
\tilde{w}_{2,2} \lambda_{1,2} & \tilde{w}_{2,2} \lambda_{2,2}
\end{pmatrix}
\approx
\begin{pmatrix}
\lambda_{1,1}^2 & \lambda_{1,1} \lambda_{2,1}
\lambda_{2,2} & \lambda_{2,2}^2
\end{pmatrix}.
$$

Normalizing this matrix by the largest diagonal elements we obtain

$$
\begin{pmatrix}
\frac{1}{\lambda_{1,1}} & 0 \\
0 & \frac{1}{\lambda_{2,2}}
\end{pmatrix}
\tilde{\Lambda}^\top \Lambda
\begin{pmatrix}
\frac{1}{\lambda_{1,1}} & 0 \\
0 & \frac{1}{\lambda_{2,2}}
\end{pmatrix}
\approx
\begin{pmatrix}
1 & \frac{\lambda_{2,1}}{\lambda_{1,1}} \\
\lambda_{2,2} & 1
\end{pmatrix} =: B.
$$

The smallest singular value $\sigma_{\min}(B)$ of the matrix $B$ is a measure for how large the off-diagonal elements are. In the special case of a diagonal matrix $B$, the smallest singular value equals 1. Our lower bound will depend on a probabilistic bound for $\sigma_{\min}(B)$, which depends on the distribution of the loading vectors. For example in the special case of i.i.d normally distributed loading elements a random element of the loading vector is $O_p(1)$, while the largest element is unbounded in the limit. Hence, the ratio is $\frac{\lambda_{2,1}}{\lambda_{1,1}} = o_p(1)$. For this special case the multi-factor case is a direct extension of the one-factor model, i.e. we can apply the one-factor result to each factor in the multi-factor model individually. Another special case is when the population loadings have sparse structures themselves. In both special cases we have $\lim_{N \to \infty} \sigma_{\min}(B) = 1$. However, in general we need to take the effect of $\sigma_{\min}(B)$ into account.

Under Assumption 6 we state the corresponding multi-factor version to Theorem 3:

**Theorem 4.** Under Assumptions 1-6 the asymptotic lower bound equals

$$
\lim_{N,T \to \infty} P(\rho \geq \rho_0) \geq \prod_{j=1}^K \left(1 - G^*_{j,m}(\tau)\right) - \lim_{N \to \infty} P(\sigma_{\min}(B) < \gamma)
$$

$$
\rho_0 = K - \frac{(1 + h(m))\sigma^2}{m^2} \sum_{j=1}^K \frac{1}{s_j u_{j,N}(\tau)},
$$

where $s = \text{diag}(s_1, s_2, \ldots, s_K)$ are the eigenvalues of $\Sigma F \Sigma$ in decreasing order, $B = [\beta_{kl}] \in R^{K \times K}$ with $\beta_{ll} = 1$ and $\beta_{kl} = \frac{1}{s_k \sum_{i=1}^m v_{i,k} v_{i,l}}$, $\sigma_{\min}(B)$ is the minimum singular value of $B$ and $0 < \gamma < 1$.

When the entries of each loading vector $V_j$ are i.i.d. we obtain a simple corollary. Here the assumptions of Lemma 1 are replaced by the assumption that for each loading vector $V_j$
the entries are i.i.d. and the largest absolute value follows asymptotically an extreme value distribution.

**Corollary 2.** Under Assumptions 1-6 and i.i.d. loadings, we have

$$\rho_0 = K - \frac{(1 + h(m))\sigma_e^2}{m} \sum_{j=1}^{K} \frac{1}{s_j(a_{j,N}z + b_{j,N})^2}$$

with the extreme value distribution $G^{*}_{j,m}(z) = e^{-\tau_j^*(z)} \sum_{s=0}^{m-1} \tau_j^*(z)^s$ and $\tau_j^*(z) = \left[1 + \xi_j \left(\frac{z-\mu_j}{\sigma_j^*}\right)\right]^{-1/\xi_j^*}$.

Inequality (13) clearly lays out what affects the generalized correlation $\rho$. Compared to Theorem 3, Theorem 4 depends on the additional parameter $\gamma$. The lower threshold $\rho_0$ decreases with $\gamma$ lowering the probability on the left-hand side in Inequality (13). On the other hand, $P(\sigma_{\min}(B) < \gamma)$ also increases with $\gamma$ decreasing the right-hand side in Inequality (13). Overall, the probabilistic lower bound gets worse with increasing $\gamma$.

The additional parameter $\gamma$ accounts for the impact of the off-diagonal terms to the trace of the matrix $\left(\frac{1}{T}\tilde{W}^T(\Lambda F^T F \Lambda^T)\tilde{W}\right)^{-1}$, a term in the asymptotic lower bound for $\rho$. For example, in the special case when the population loading $V = \Lambda H$ is “sparse”, i.e. it satisfies the non-overlapping condition, the off-diagonal terms in $\frac{1}{T}\tilde{W}^T(\Lambda F^T F \Lambda^T)\tilde{W}$ converge to 0; $\sigma_{\min}(B) = 1$ and $P(\sigma_{\min}(B) < \gamma) = 0$ for any $\gamma < 1$. Thus, the probabilistic lower bound only depends on the diagonal entries of $\frac{1}{T}\tilde{W}^T\Lambda F^T F \Lambda^T\tilde{W}$, each of which can be characterized by the extreme value distribution of the corresponding column in $V = \Lambda H$.

For the general case, the correction probability $P(\sigma_{\min}(B) < \gamma)$ can be estimated with a bootstrap applied to the estimated loadings $\hat{\Lambda}$. In special cases for the loading distribution there exits analytical solutions, e.g. if the entries of the population loading vectors $\Lambda_k$ are i.i.d. normal, it holds that $P(\sigma_{\min}(B) < \gamma) = 0$ and we can neglect the correction term.

The same comparative statics hold as in the single-factor case. For a given lower probability bound in Corollary 2, e.g. 95%, the threshold $\rho_0$ is: (1) decreasing in the noise level $\sigma_e^2$; (2) decreasing in the cross-section noise dependence level $h(m)$; (3) increasing in the signal level $s_j$; (4) increasing with $z$ and increasing with $N$ if $a_{j,N}$ and $b_{j,N}$ are among the examples listed in the one-factor case.

In the second, more general case we first rotate the loadings before thresholding them. The assumption that the sparse and normalized vectors of appropriately rotated loadings are orthonormal is more general than imposing this assumption directly on the population loadings. In more detail, we assume that there exists an orthonormal rotation matrix $P$ such that the matrix $V^P = VS^{1/2}P$ satisfies an “approximate non-overlapping” assumption, formally defined in Assumption 7. It turns out that for this case the asymptotic theory
is more convenient to state for the matrix $V S^{1/2}$, where $S$ is the diagonal matrix of the eigenvalues of $\Sigma F \Sigma \Lambda$ in decreasing order and $V = \Lambda H \in \mathbb{R}^{N \times K}$ as before.

**Assumption 7** (Approximate non-overlapping). For the matrix $V^P = VS^{1/2}P$, there are at least $m$ entries in $|V_j^P|$ that satisfy $\max_{i,k \neq j} |v_{i,k}^P / v_{i,j}^P| < c$ for all $j$, where $c$ satisfies $0 < c < 1/(2\sqrt{K(K-1)})$ when $K = 2$ and $0 < c < (\sqrt{1 + (K-2)/\sqrt{K(K-1)}} - 1)/(K-2)$ when $K > 2$.

Assumption 7 is much weaker than the “non-overlapping” assumption in that (1) it allows $V^P$, as well as $V$, to be dense (all entries can be nonzero) (2) it requires sufficiently many large absolute values in different columns of $V^P$ are taken in disjoint rows instead of $V$ ($V^P$ may still have some large absolute values taken in the same rows). Note, that the $m$ values that we choose to construct the proximate factors are not necessarily the largest loading elements. Here we allow for example to take the $m + 1$ largest loading element of the rotated loading vector $V_k$ as long as this entry is relatively small for the other vectors $V_j$ with $j \neq k$.

Let the sparse factor weights under Assumption 7 be $\tilde{W}^P$. The weight vector corresponding to each factor has $m$ nonzero entries. They are the largest in $\hat{V}^P = \hat{\Lambda}P$ satisfying $\max_{i,k \neq j} |\hat{v}_{i,k}^P / \hat{v}_{i,j}^P| < c$ and are standardized to have $\|\tilde{W}^P_j\|_2 = 1$.\textsuperscript{22} The proximate factors are obtained from the least-square regression

$$\tilde{F}^P = X^T \tilde{W}^P ((\tilde{W}^P)^T \tilde{W}^P)^{-1}$$ \hspace{1cm} (14)

As a result, a probabilistic lower bound of

$$\rho = tr \left( (F^T F/T)^{-1}(F^T \tilde{F}^P / T)((\tilde{F}^P)^T \tilde{F}^P / T)^{-1}((\tilde{F}^P)^T F / T) \right)$$

is the sum of $K$ squared correlations between each rotated population factor and corresponding proximate factor $\tilde{F}^P$ multiplied by an adjustment term depending on $c$. The following theorem makes the idea precise.

**Theorem 5** (Rotate and threshold). Under Assumptions 1-4 and 7, let $\tilde{v}_{(m),j}^P$ be the $m$-th order statistic of the entries in $|V_j^P|$ and assume that the cumulative density function of $\tilde{v}_{(m),j}^P$

\textsuperscript{22}The standardized vectors are given by

$$\tilde{W}^P = \begin{bmatrix} \tilde{V}_1^P \otimes M_1 & \tilde{V}_2^P \otimes M_2 & \ldots & \tilde{V}_K^P \otimes M_K \end{bmatrix}.$$
is continuous. Then for a given threshold $0 < \rho_0 < K$ and a fixed $m$, we have

$$
\lim_{N,T \to \infty} P(\rho > \rho_0) \geq \lim_{N \to \infty} P\left( \sum_{j=1}^{K} \frac{1}{(\tau_{(m),j}^P)^2} < \frac{m(1 - \gamma)(K - \rho_0)}{(1 + h(m))\sigma_e^2} \right),
$$

where $\gamma = c(2 + c(K - 2))(K(K - 1))^{1/2}$. 

The choice of the rotation matrix $P$ creates a trade-off between choosing large factor weights with more signal information and selecting factor weights that are close to orthogonal. The issue is again that in general $(\hat{W}^P)^\top V^P (V^P)^\top \hat{W}^P$ is not a diagonal matrix. Relative Weyl’s Theorem provides bounds of the eigenvalues of $(\hat{W}^P)^\top V^P (V^P)^\top \hat{W}^P$. The relative difference between the eigenvalues and diagonal entries can be bounded by the ratio of the sum of off-diagonal to diagonal entries. This ratio can be controlled by $\gamma$ which is defined in Theorem 5 and is increasing in $c$. The larger the $c$, the wider the bounds of the relative differences, therefore the smaller the lower bound. From this perspective, we would like to select a $c$ as small as possible. However, if $c$ is small, there are fewer entries that can satisfy $\max_{j,k \neq j} |v_{i,k}^P/\hat{S}_{i,j}^{1/2}| < c$ decreasing $\tau_{(m),j}^P$. From this perspective, we would like to select $c$ as large as possible. A good $c$ can balance these two aspects and maximize the asymptotic probabilistic lower bound in Equation (15). The signal level $S$ is part of $V^P$ from Equation (15), so that the larger the signal level, the larger the lower bound. Moreover, the larger the noise level, $\sigma_e^2$ and cross-section dependence level of errors $h(m)$, the smaller the lower bound. These properties are aligned with those in previous theorems.

An alternative method is to first threshold $\hat{\Lambda}$ to get $\hat{W}$ and then multiply $\hat{W}$ by $\hat{S}^{1/2} P$, so that the nonzero entries in different columns of $\hat{W} \hat{S}^{1/2} P$ do not overlap. The generalized correlation is the same whether $\hat{W}$ is multiplied by $\hat{S}^{1/2} P$ or not. In practice, this method is more attractive because we do not need to find the $P$ and simply perform the three steps of our proposed method in the model setup section. This method is shown to work very well in our empirical applications and simulations. However, the theoretical results for an analytical bound require very strong assumptions. First, we assume $P$ exists such that $\hat{W} \hat{S}^{1/2} P$ satisfies Assumption 7 even if $\hat{W}$ might violate Assumption 7. This assumption is hard to be satisfied when $m \gg K$. The number of nonzero entries is uncertain for each column in $\hat{W} \hat{S}^{1/2} P$, which significantly complicates the distribution of $\rho$. Second, the distribution of the largest entries of the rotated non-sparse population loadings can be different from the distribution of the rotated sparse factor weights. Therefore, additional assumptions need to be made to obtain the theoretical results which are tedious due to the nature of $\rho$’s complicated distribution. We have not included these theoretical results here but they are available upon request.
3.5 Comparison with Sparse PCA

Our method is closely related to using Lasso (Tibshirani 1996) or Elastic Net (Zou and Hastie 2005) to sparsify loadings estimated from PCA. Most papers about estimating sparse principal components are more or less related to the method proposed by Zou et al. (2006). Zou et al. (2006) estimate sparse loadings by adding $\ell_1$ penalty terms (and $\ell_2$ penalty terms) to the objective function (2) yielding the formulation

$$\hat{(F, \Lambda)} = \arg\min_{F, \Lambda} \sum_{i=1}^{N} \|x_i - FA^\top x_i\|^2_2 + \sum_{j=1}^{K} \alpha_j \|\Lambda_j\|_1 + \gamma \sum_{j=1}^{K} \|\Lambda_j\|_2^2.$$  \hspace{1cm} (16)

subject to $F^\top F = I_{K \times K}$ \hspace{1cm} (17)

All three methods (proximate factors, Lasso and Elastic Net) are easy to implement. However, our method is different from Lasso and Elastic Net in three aspects. First, our method does not impose the sparsity assumption on the loadings. The sparse factor weights are used to construct the proximate factors. However, the loadings to the proximate factors are in general non-sparse. Second, even though all methods have tuning parameters, these parameters work differently. In our method $m$ is the number of nonzero entries in each factor weight vector. Although $\eta$ in Lasso and Elastic Net controls the sparsity of loadings, $\eta$ cannot control the exact number of nonzero entries in individual loadings. Third, the thresholding in our approach is essentially a variable selection without changing the proportion of the largest weights, while the shrinkage in sparse PCA selects and rescales the largest loadings. It is well-known that a Lasso estimator is biased because it scales down the population parameters. A similar phenomena occurs with sparse PCA. For a simple case we can show in Proposition 3 below that this downscaling leads to a larger bias than our method.

We want to point out that the comparison between sparse PCA and our proximate factors is not the key element of this paper. Our insight that factor weights are different from factor loadings can also be taken into account with sparse PCA methods, i.e. the sparse PCA loadings can be used as factor weights and the loadings are obtained from a second stage regression. Our simulations and empirical results suggest that even in this case the proximate factors perform slightly better than sparse PCA. The more important takeaway is that for our method we can provide an asymptotic inferential theory which is missing for sparse PCA and that sparse PCA with sparse loadings does not work well in the data sets that we consider.

23The number of non-zero elements in all loadings for the Lasso estimator is monotonically decreasing in the parameter $\eta$. Hence, there is a one-to-one mapping between the level of sparsity of the whole loading matrix and the $\ell_1$ penalty weight. However, except for special cases $\eta$ cannot control the sparsity of a specific loading vector.
We consider now the special case of a one factor model with cross-sectionally i.i.d. distributed errors. For sparse PCA with \( \ell_1 \) penalty similar to Jolliffe et al. (2003) we can map this estimator into our framework. In more detail, we estimate the loadings \( \tilde{\Lambda} \) by minimizing \( \| X - \Lambda \hat{F}^\top \|^2_F + \alpha \| \Lambda \|_1 \) and use them as weights to construct the sparse PCA factors \( \tilde{F} = X^\top \tilde{\Lambda} (\tilde{\Lambda}^\top \tilde{\Lambda})^{-1} \). The difference in the generalized correlation between the our method and sparse PCA is

\[
\Delta \rho = \text{tr} \left( (F^\top F/T)^{-1} (F^\top \tilde{F} / T)(\tilde{F}^\top \tilde{F} / T)^{-1} (\tilde{F}^\top F / T) \right) \\
- \text{tr} \left( (F^\top F/T)^{-1} (F^\top \bar{F} / T)(\bar{F}^\top \bar{F} / T)^{-1} (\bar{F}^\top F / T) \right)
\]

(18)

**Proposition 3.** Assume a one factor model and \( \frac{1}{T} e_i^\top e_i \to \sigma^2_e \). The tuning parameters \( m \) respectively \( \alpha \) are such that proximate factors and sparse PCA select the same number of non-zero elements. For \( N, T \to \infty \), we have with probability one \( \Delta \rho \geq 0 \).

Proposition 3 states that our method approximates the population factors at least as good as sparse PCA. Although Proposition 3 is stated in a restrictive setting, we show in simulations that \( \tilde{F} \) has a higher generalized correlation with \( F \) even when the assumptions in Proposition 3 do not hold.

### 3.6 Choice of Non-Zero Elements \( m \)

We use two different ways to select the number of non-zero elements \( m \) in the proximate factors. First, we use the theoretical results to choose \( m \) such that we have a lower bound of the target average generalized correlation, typically 95% or 97.5%. This requires either knowledge or estimation of the parameters in the extreme value distribution. Second, we use a completely data-driven approach. We estimate the average generalized correlation between the factors estimated by PCA and our proximate factors and set \( m \) to achieve a target correlation. This second approach can be implemented on a training data set to estimate the factor weights and the non-zero elements \( m \), then used out-of-sample on a test data. We pursue this approach in our empirical analysis.

### 4 Simulation

We test the model predictions in various simulations. Our baseline model is

\[ X_{it} = \lambda_i^T f_t + e_{it} \]
where $\lambda_i \sim \mathcal{N}(0, I_K)$, $f_t \sim \mathcal{N}(0, \Sigma_F)$, $e_{it} \sim \mathcal{N}(0, 1)$.

In the multi-factor case, we assume loadings are “sparse” and “non-overlapping”. Thus, we can follow the same steps to calculate the proximate factors $\hat{F}$ in both cases. We calculate the generalized correlation $\rho$ between $F$ and $\hat{F}$. For each set of parameters $N$, $T$, $\Sigma_F$ and $m$, we run 1000 Monte-Carlo simulations and calculate the percentage that $\rho$ is greater than a fixed threshold $\rho_0$ and use it as the empirical probability of $P(\rho \geq \rho_0)$.

4.1 One-factor Case

We start with the case of only one factor and study the properties of $p = 1-G_{1,m}(\tau)$ as a lower bound for $P\left(\rho \geq \frac{\sigma_{1i}^2 u_{1,N}(\tau)}{(1 + h(m)) \sigma_2^2 + m \sigma_{j1}^2 u_{j,N}^2(\tau)}\right)$ in Theorem 3. We require the squared correlation between $F$ and $\hat{F}$ to be at least 0.95 or equivalently the absolute correlation to be larger than 0.975. First, we solve for $\tau$ from $\rho_0 = \frac{\sigma_{j1}^2 u_{j,N}(\tau)}{(1 + h(m)) \sigma_2^2 + m \sigma_{j1}^2 u_{j,N}^2(\tau)} = \frac{\sigma_{1i}^2 u_{1,N}(\tau)}{\sigma_2^2 + m \sigma_{j1}^2 u_{j,N}^2(\tau)} = 0.95$, which we then use to calculate $\rho^2$. Figure 1 shows how $P(\rho \geq \rho_0)$ and $p$ change with different nonzero entries $m$, $T$ and signal-to-noise ratio $\sigma_{j1}/\sigma_e$. In all the three subplots with the signal-to-noise ratios 0.8, 1.0 and 1.2, $p$ is very close to $P(\rho \geq \rho_0)$, especially when $T$ is large ($T = 200$). Equation (11) has an $o_p(1)$ term and this term decreases with $T$. Thus, as $T$ increases, $p$ becomes uniformly smaller than $P(\rho > \rho_0)$ with different $m$. Notice that as $m$ increases to 10, $P(\rho \geq \rho_0)$ becomes very close to 1, meaning that $\hat{F}$ has 97.5% correlation with $F$ but is based on only 10% of all observations. Moreover, the larger the signal-to-noise ratio, the larger $P(\rho \geq \rho_0)$ and $p$. The pattern is more obvious when the number of nonzero elements $m$ is small. In addition, as seen in all three subplots, as $m$ increases, both $P(\rho \geq \rho_0)$ and $p$ increase. Figure 2a shows that, as $N$ increases, both $P(\rho \geq \rho_0)$ and $p$ increase to 1 with different $T$ for the fixed number of nonzero elements $m = 4$ and the threshold $\rho_0 = 0.95$. Since $\lambda_i$ is normally distributed it has unbounded support and the $m$ selected cross-section units have strong enough signals to obtain a squared correlation between $F$ and $\hat{F}$ of at least 95%.

4.2 Multi-factor Case

Next, we consider the case of two factors, i.e. $K = 2$. We test $p = (1-G_{1,m}(\tau))(1-G_{2,m}(\tau))$ as a lower bound for $P\left(\rho \geq 2 - \frac{(1 + h(m)) \sigma_2^2}{m} \sum_{j=1}^{2} \frac{1}{s_j^2 u_{j,N}^2(\tau)}\right)$ based on Theorem 4. We fix

24 We also simulate data with time-series dependence or with cross-section dependence in the errors. The results are very similar to those using data with i.i.d errors and are available upon request.

25 Note that in one-factor case, $\sigma_{j1} = (\Sigma_F)^{1/2}$ and $\sigma_e = 1$. Since $\lambda_i \sim \mathcal{N}(0,1)$, we have $u_{1,N}(\tau) = a_{1,N} \left(\mu^* + \sigma^* \left(\frac{\tau^* - 0.5}{\sigma^*}\right)\right) + b_{1,N}$, where $a_{1,N} = \Phi^{-1}(1 - 1/(2N))$ and $b_{1,N} = \Phi^{-1}(1 - 1/(2N))$ are the location and scale of a folded (absolute) standard normal.
Figure 1: One-factor model. Dots are empirical probabilities $P(\rho \geq \rho_0)$ with different $T$, where $\rho_0 = 0.95$. The curve is $1 - G_{1,m}(\tau)$ from equation (11) in Theorem 3. ($N = 100$, $\sigma_e = 1$).

Figure 2: (a) One-factor model. Dots are empirical probabilities $P(\rho \geq \rho_0)$ with different $T$, where $\rho_0 = 0.95$. The curve is $1 - G^*(z)$ from equation (11) in Theorem 3 ($\sigma_f = 1.0$, $\sigma_e = 1$); (b) Two-factor model. Dots are empirical probabilities $P(\rho > \rho_0)$ with different $T$, where $\rho_0 = 1.9$. The curve is $(1 - G_{1,m}(\tau))(1 - G_{2,m}(\tau))$ in Equation (13) in Theorem 4 ($m = 4$, $\sigma_f = [1.2, 1.0]$, $\sigma_e = 1$).
Figure 3: Two-factor model. Dots are empirical probabilities $P(\rho > \rho_0)$ with different $T$, where $\rho_0 = 1.9$. The curve is $(1 - G_{1,m}(\tau))(1 - G_{2,m}(\tau))$ in Equation (13) in Theorem 4 $(N = 100, \sigma_e = 1)$. 

\[\rho_0 = 2 - \frac{(1+h(m))\sigma_e^2}{m} \sum_{j=1}^{2} \frac{1}{\lambda_j^2} \approx 1.9.\] As \(\rho_0\) is increasing in \(\tau\), we can use the bisection method to solve for \(\tau\) and then calculate \(p\). We assume factors are independent, i.e. $\Sigma_F = \text{diag}(\sigma^2_{f_1}, \sigma^2_{f_2})$.

The signals equal $s_j = \sigma^2_j$ for $j = 1, 2$. Figure 3 shows how $P(\rho \geq \rho_0)$ and $p$ change with different nonzero entries $m$, $T$ and signal-to-noise ratios $\sigma_{f_1}/\sigma_e$ and $\sigma_{f_2}/\sigma_e$. We generate three subplots with three pairs of signal-to-noise ratios $[1.0, 0.8]$, $[1.2, 1.0]$ and $[1.5, 1.2]$ from left to right. Similar to the one-factor case, the gap between $p$ and $P(\rho \geq \rho_0)$ is smaller with stronger signal-to-noise ratios and more cross-section units in the proximate factors. $P(\rho \geq \rho_0)$ becomes very close to 1 with moderately strong signal-to-noise ratios and around 10 nonzero entries in the sparse loadings, meaning that each $\tilde{F}$ has around 97.5% correlation with the corresponding $F$ but is based on only 10% of all observations. The patterns of how $P(\rho > \rho_0)$ and $p$ change with signal-to-noise ratios, number of nonzero entries in the two-factor case are the same as for one factor. Similar to the one-factor case, Figure 2b shows that, as $N$ increases, both $P(\rho \geq \rho_0)$ and $p$ increase to 1.

### 4.3 Comparison with Sparse PCA

We compare the estimation of the factors, loadings and the common component of our methods with various versions of sparse PCA. We have already shown theoretically that in the special case of a one-factor model with homoskedastic errors, the proximate factors provide a better estimator than sparse PCA. We consider now the more general setting of multiple factors with dependent and heteroscedastic errors. For the various estimators we calculate the generalized correlation of the estimated factors and loadings with the population values and also calculate the root-mean-squared error (RMSE), i.e. \[\sqrt{\frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} \left( X_{it} - \hat{X}_{it} \right)^2},\]

\[26\text{Since } \lambda_i \sim N(0, J_2), a_{j,N} = \frac{1}{2N\phi(b_{j,N})} \text{ and } b_{1,N} = \Phi^{-1}(1 - 1/(2N)) \text{ are the location and scale of } \lambda_{i,j} \text{ for } j = 1, 2.\]
where \( \hat{X}_{it} \) is the common component predicted by the models. We report the in-sample and out-of-sample results. For the first case we estimate the parameters on the training data set and calculate the goodness-of-fit measures on it. For the second case we use the factor portfolio weights estimated on the training data set to construct the factors and loadings on the test data set.

Motivated by our empirical application we simulate a \( K = 5 \) factor model with weakly correlated heteroskedastic errors:

- Heteroskedasticity: \( e_{it} = \sigma_i v_{it}, \sigma_i \sim U(0.5, 1.5) \) and \( v_{it} \sim N(0, 1) \)
- Cross sectional dependence: \( e_t \sim N(0, \Sigma_e) \), where \( \Sigma_e = (c_{ij}) \in \mathbb{R}^{N \times N} \) with \( c_{ij} = 0.5^{|i-j|} \)

We label our method PPCA (proximate PCA). Sparse PCA (SPCA) with an \( \ell_1 \) penalty estimates sparse factor weights and sparse loadings. We modify sparse PCA (SPCA (mod)) by using the sparse factor weights and obtain non-sparse loadings in a second stage regression. This is similar to our approach but with \( \ell_1 \) shrinkage instead of hard thresholding. In order to make the approaches comparable, we choose the number on non-zero elements \( m \) for PPCA s.t. it coincides with the number of non-zero elements for SPCA for a given \( \ell_1 \) penalty \( \alpha \).

Figure 4 reports the generalized correlations for factors and loadings and RMSE for the different methods. On both test and training data set PPCA has higher correlations with the population factors and smaller RMSE than both versions of sparse PCA.\(^{27}\) Interestingly PPCA provides actually a fit as good as PCA without thresholding. It is striking that sparse PCA with sparse loadings has a substantially lower correlation with the true loadings and twice the error compared to its modified version with non-sparse loadings. Overall the simulation results confirm that the theoretical result for a one-factor model with homoskedastic errors extends to a more general setup.

## 5 Empirical Application

In this section, we apply our method to two significantly different datasets, 370 single-sorted stock portfolios and 128 macroeconomic variables. We compare the proximate factors \( \tilde{F} \) with the PCA factors \( \hat{F} \) by two metrics, the generalized correlation \( \rho \) between \( \tilde{F} \) and \( \hat{F} \) and the proportion of variation explained by \( \tilde{F} \) and \( \hat{F} \). In both datasets, proximate factors constructed from our approach with 5-10% of cross-section observations are very close to the

\(^{27}\)Note that SPCA and its modified version give the same factors but different loadings. The loadings for SPCA on the test data set are the same as on the training data set. The loadings of PPCA and the modified SPCA are different on the test and training data set as they are estimated in a second stage regression.
Figure 4: Generalized correlations for factors and loadings and RMSE for proximate PCA (PPCA), sparse PCA (SPCA) and modified sparse PCA with second stage loading regression. \(\alpha\) is the \(\ell_1\) penalty for SPCA with \(m\) chosen accordingly. \(N = 100, T = 100, K = 5\) conventional PCA factors. As proximate factors are composed of only a few cross-section observations, we can find economically meaningful labels for each latent factor.

5.1 Financial Single-Sorted Portfolios

We study the monthly returns of 370 portfolios sorted in deciles based on 37 anomaly characteristics from 07/1963 to 12/2016. For each characteristic, we have ten decile-sorted portfolios that are updated yearly and are linear combinations of returns of U.S. firms in CRSP. These portfolios are compiled by Kozak et al. (2017) following the standard anomaly definitions in Novy-Marx and Velikov (2015). These 37 anomaly characteristics are listed in Table 3 in the Appendix. Each of these portfolios can be interpreted as an actively managed portfolio based on a signal that has been shown to be relevant for the risk-return tradeoff. In contrast to individual stock data these managed portfolios are more stationary and can be modeled by a stable factor structure (Lettau and Pelger (2018b)).

In Figure 5, we calculate the generalized correlation \(\rho\) between estimated PCA and proximate factors and compare the proportion of variation explained by these factors for different number of factors \(K\). We normalize the generalized correlation by the number of factors. The closer \(\rho/K\) to 1, the better the sparse factors approximate the non-sparse PCA factors. The more portfolios we include in the proximate factors, the larger the correlation \(\rho\) and the amount of variation explained. When the number of nonzero entries \(m\) in each sparse loading is around 20, corresponding to around 5% of the cross-section units, the ratio is close to 0.9, implying that the average correlation between each proximate factor in \(\tilde{F}\) and

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\(28\) We thank Kozak et al. (2017) for allowing us to use their data.  
\(29\) Kozak et al. (2017) use a set of 50 anomaly characteristics. We use 37 of those characteristics with the longest available cross-sections. The same data is studied in Lettau and Pelger (2018b). They argue that this data can be modeled by a stable factor structure.
Figure 5: Financial single-sorted portfolios: Generalized correlation between $\hat{F}$ and $\tilde{F}$ normalized by $K$ and proportion of variance explained by $\hat{F}$ and $\tilde{F}$ as a function of non-zero loading elements $m$, where $K$ varies from 3 to 7. ($N = 370, T = 638$)

each estimated factor in $\hat{F}$ is around 95%. Since $\hat{F}$ explains the maximum variation among all matrices in $\mathbb{R}^{T \times K}$, the proportion of variation explained by $\hat{F}$ is strictly less. However, the variation explained by $\tilde{F}$ becomes very close to that by $\hat{F}$ when $\tilde{F}$ is constructed from the 10 cross-section units with the strongest signals. Lettau and Pelger (2018a) argue that 5 factors describe this data set well. Hence, we focus now on a 5-factor model and try to interpret these factors. Table 1 shows the generalized correlations for each of the five factors with their proximate version. As by construction, the PCA factors are uncorrelated, the individual generalized correlations correspond to the $R^2$s from a linear regression of each PCA factor on the five proximate factors. Even with only 10 portfolios, we can capture each of the latent factors very well, in particular the first two factors. 30 portfolios are sufficient to almost perfectly replicate the latent factors.$^{30}$

Table 1: Financial single-sorted portfolios: Generalized correlation between each $\hat{F}_j$ and all $\tilde{F}$ for $K = 5$. These generalized correlations correspond to $R^2$ from a regression of each $\hat{F}_j$ on all $\tilde{F}$.

$^{30}$As many investors trade on factors, trading on proximate factors can also significantly reduce trading costs.
We study each of the five proximate factors in more detail. Figure 7 provides a clear picture of the assets used to construct the fourth factor, labeled as the momentum factor. We use the fourth factor as an example and present the figures for the other factors in the Appendix. First, the fourth proximate factor is only composed of five strategies related to momentum, which are Industry Momentum, 6-month Momentum, 12-month Momentum, Value Momentum and Value Momentum Profitability. Second, this factor is a long-short factor, i.e., it has negative weights on the smallest decile portfolios and positive weights on the largest decile portfolios. Most financial factors are constructed as long-short factors to capture the difference between the extreme characteristics. For comparison we also show the portfolio weights for the fourth PCA factor in Figure 8. The pattern suggests that the fourth factor is related to momentum, but our proximate factor provides the argument that the information for this factor is almost completely captured by momentum portfolios.

Figure 6 shows the main composition of each latent factor based on the anomaly strategies. It provides a first intuition for interpreting each latent factor. Based on Figures 13 to 16 we assign the following labels to the other latent factors: The first factor is a “long”-only market factor. The second factor loads on categories such as dividend/price, earning/price, investment capital and asset and sales growth suggesting a “value” interpretation. The third factor loads strongly on price, value categories and size and hence we label it as a price-value-size factor. The last factor is clearly an asset turnover-profitability factor.
Figure 7: Financial single-sorted portfolios: Portfolio weights of 4th proximate factor. The sparse loading has 30 nonzero entries. The full names of these 37 anomaly characteristics are listed in Table 3 in the Appendix.

Figure 8: Financial single-sorted portfolios: Portfolio weights of 4th PCA factor.

Last but not least, Figure 9 compares proximate factors with sparse PCA using the same metrics as in Section 4.3. As the population factors and loadings are unknown, we compare the proximate and sparse factors and loadings with the non-sparse PCA factors and loadings. The first half of the time-series serves as the training and the second half as the test data set. The number of non-zero elements $m$ is again set such that it equals the number of non-zero elements for SPCA for a given $\ell_1$-lasso penalty $\alpha$. The main takeaways are as follows: First, sparse PCA with sparse loadings leads to substantially worse results than proximate factors. Second, the proximate factors are almost perfectly correlated with the PCA factors and result in the same RMSE. Third, the modified SPCA performs better than its version with sparse loadings, but not as good as our method.
Figure 9: Financial single-sorted portfolios: Generalized correlations for factors and loadings and RMSE for proximate PCA (PPCA), sparse PCA (SPCA) and modified sparse PCA with second stage loading regression. \( \alpha \) is the \( \ell_1 \) penalty for SPCA with \( m \) chosen accordingly.

5.2 Macroeconomic Data

We study macroeconomic data with 128 monthly U.S. macroeconomic indicators from 01/1959 to 02/2018 as in (McCracken and Ng, 2016). This data is from the Federal Reserve Economic Data (FRED) and is publicly available. The 128 macroeconomic indicators can be classified into 8 groups: 1. output and income; 2. labor market; 3. housing; 4. consumption, orders and inventories; 5. money and credit; 6. interest and exchange rates; 7. prices; 8. stock market (McCracken and Ng, 2016). Macroeconomic datasets have been successfully analyzed with PCA factors for forecasting and estimation of factor augmented regressions (Stock and Watson, 2002b; Boivin and Ng, 2005). McCracken and Ng (2016) estimate 8 factors in this macroeconomic data.

Figure 10 plots the generalized correlation between \( \tilde{F} \) and \( \hat{F} \) normalized by \( K \) and the proportion of variance explained with different number of factors. When the number of nonzero entries \( m \) in each sparse factor weight vector is around 10, which is less than 8% of all macroeconomic variables, the ratio of \( \rho \) to \( K \) is close to 0.9, which corresponds to an average correlation of 0.95%. The amount of variation explained by \( \tilde{F} \) and \( \hat{F} \) also indicate \( \tilde{F} \) consisting of around 15 macroeconomic variables can approximate \( \hat{F} \) very well.

In the following, we study the 8-factor model in more details. Table 2 shows the generalized correlations for each factor which correspond to the \( R^2 \) in regressions of each PCA factor on all proximate factors. The first five factors can be captured well by proximate factors with 10 macroeconomic variables in each proximate factors. The sixth to eighth factors are weaker, but they can be reasonably captured well by proximate factors with around 20 variables.

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31 This dataset is updated in a timely manner and is available at https://research.stlouisfed.org/econ/mccracken/fred-databases/
Figure 10: Macroeconomic data: Generalized correlation between $\tilde{F}$ and $\hat{F}$ normalized by $K$ and proportion of variance explained by $\tilde{F}$ and $\hat{F}$, where $K$ varies from 4 to 20. ($N = 128$, $T = 707$)

<table>
<thead>
<tr>
<th>$m$</th>
<th>$\hat{F}_1$</th>
<th>$\hat{F}_2$</th>
<th>$\hat{F}_3$</th>
<th>$\hat{F}_4$</th>
<th>$\hat{F}_5$</th>
<th>$\hat{F}_6$</th>
<th>$\hat{F}_7$</th>
<th>$\hat{F}_8$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>0.953</td>
<td>0.959</td>
<td>0.949</td>
<td>0.953</td>
<td>0.961</td>
<td>0.799</td>
<td>0.833</td>
<td>0.767</td>
</tr>
<tr>
<td>15</td>
<td>0.967</td>
<td>0.970</td>
<td>0.958</td>
<td>0.956</td>
<td>0.964</td>
<td>0.857</td>
<td>0.867</td>
<td>0.837</td>
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<td>20</td>
<td>0.977</td>
<td>0.974</td>
<td>0.957</td>
<td>0.963</td>
<td>0.961</td>
<td>0.905</td>
<td>0.919</td>
<td>0.891</td>
</tr>
<tr>
<td>25</td>
<td>0.983</td>
<td>0.980</td>
<td>0.961</td>
<td>0.979</td>
<td>0.973</td>
<td>0.937</td>
<td>0.943</td>
<td>0.929</td>
</tr>
</tbody>
</table>

Table 2: Macroeconomic data: Generalized correlation between each $\hat{F}_j$ and $\tilde{F}$, where $K = 8$. These generalized correlations correspond to $R^2$ from a regression of each $\hat{F}_j$ on all $\tilde{F}$. 

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Figure 11: Macroeconomic data: 8-factor model, each sparse factor weight vector in $\tilde{W}$ has 10 nonzero entries. Values in this figure represent the number of nonzero entries in a particular group for a particular sparse factor weight vector. The 8 groups are: 1. output and income; 2. labor market; 3. housing; 4. consumption, orders and inventories; 5. money and credit; 6. interest and exchange rates; 7. prices; 8. stock market.

Figure 11 shows a clear interpretation of the statistical factors. When each hard-thresholded loading has 10 nonzero entries, we see a strong pattern in the composition of each latent factor from the sparse factor weights and can label each factor. The first factor is a labor and productivity factor; the second factor is a price factor; both third and fourth factors are interest and exchange rate factors; the fifth factor is a housing and rate factor; the sixth factor is a labor and stock market factor; the seventh factor is a productivity and stock market factor; the eighth factor is a labor and rate factor. Macroeconomic variables in groups of output, income, labor market and prices are the main compositions of the first and second factors. The third, fourth, fifth and eighth latent factors are mainly composed of macroeconomic variables in the group of interest and exchange rates. Variances explained by latent factors are in decreasing order. Thus, variables in groups of output, income, labor market, prices, interest and exchange rates explain most of the variation in the whole dataset.

Finally, Figure 12 compares proximate factors with sparse PCA with the same metrics as before. We split the data set into half for training and testing. The findings confirm our previous results. SPCA performs worse among all dimensions. Proximate factors give essentially the same results as standard PCA as long as the sparsity level is not too high. Surprisingly, the modified SPCA can have slightly smaller RMSE out-of-sample than standard PCA and hence also the proximate factors. Overall the results justify why we can use the proximate factors as a replacement for standard PCA factors.
6 Conclusion

In this paper, we propose a method to construct proximate factors that consist of only a small number of cross-section units and can approximate latent factors well. These proximate factors are usually much easier to interpret than PCA factors. The closeness between proximate factors and latent factors is measured by the generalized correlation. We provide an asymptotic probabilistic lower bound for the generalized correlation based on extreme value theory. This lower bound explains why proximate factors are close to latent factors and provides guidance on how to chose the sparsity in the proximate factors. Simulations verify that the lower bound closely approximates the exceedance probability of the generalized correlation, especially in the most relevant case of an exceedance probability close to 1. The proximate factors have non-sparse loadings which are consistent estimates of the true population loadings. Empirical applications to two very different datasets, financial single-sorted portfolios and macroeconomic data, show that proximate factors consisting of 5-10% of the cross-section units can approximate latent factors well. Using these proximate factors, we provide a meaningful interpretation for the latent factors in our datasets.

References


Martin Lettau and Markus Pelger. Factors that fit the time series and cross-section of stock returns. 2018b.


Supplementary Appendix to
Interpretable Proximate Factors for Large Dimensions

A Extreme Value Theory for Dependent Data

Lemma 1. Suppose that $\{|\lambda_{i,1}|\}$ is a strictly stationary sequence which satisfies the strong mixing condition. Let $|\lambda_{(m),1}|$ be the $m$-th largest in $|\lambda_{i,1}|$ for all $i$,

$$
\pi_{1,N}(i; \tau) = P \left[ \sum_{j=1}^{r_N} \mathbb{1} (|\lambda_{i,1}| > u_{1,N}(\tau)) = i \right]
$$

for some normalizing functions $u_{1,N}(\tau)$ and sequence $\{r_N\}$ that satisfies

$$
N/r_N \to \infty, e^{N/r_N} \alpha(l_N) \to 0, \text{ and } e^{N/r_N} l_N/N \to 0
$$

where $\{l_N\}$ is any sequence that satisfies $l_N/N \to 0$, $\alpha(l_N) \to 0$ and $\alpha(\cdot)$ is the mixing function of the strong mixing condition which holds for $\{|\lambda_{i,1}|\}$. Let the $l$-fold convolution of $\pi_N(\cdot; \tau)$ be

$$
\pi_1^l(i; \tau) = \begin{cases} 0, & i < l \\ \sum_{i_r \geq 1, 1 \leq r \leq l} \pi_{1,N}(i_1; \tau) \cdots \pi_{1,N}(i_l; \tau), & i \geq l. \end{cases}
$$

If for some $\tau > 0$, $\pi_{1,N}(i; \tau)$ converges to some $\pi_1(i)$ for $1 \leq i \leq m - 1$ which is independent of $\tau$, then $P(|\lambda_{(m),1}| \leq u_{1,N}(\tau))$ converges for each $\tau > 0$ and

$$
G_{1,m}(\tau) = \lim_{N \to \infty} P(|\lambda_{(m),1}| \leq u_{1,N}(\tau)) = e^{-\tau} \left[ 1 + \sum_{l=1}^{m-1} \frac{\tau^l}{l!} \left( \sum_{i=l}^{m-1} \pi_1^l(i) \right) \right] \tag{19}
$$

where

$$
\pi_1^l(i) = \begin{cases} 0, & i < l \\ \sum_{i_r \geq 1, 1 \leq r \leq l} \pi_1(i_1) \cdots \pi_1(i_l), & i \geq l. \end{cases} \tag{20}
$$

Conversely, if $P(|\lambda_{(m),1}| \leq u_{1,N}(\tau))$ converges for each $\tau > 0$, then for any $\tau > 0$ and $1 \leq i \leq m - 1$ the probability $\pi_{1,N}(i; \tau)$ converges to some $\pi_1(i)$ which is independent of $\tau$ and the limit of $P(|\lambda_{(m),1}| \leq u_{1,N}(\tau))$ is the same as (19).

Note that the asymptotic distribution of $|\lambda_{(m),1}|$ is independent of the specific choice of sequences $\{l_N\}$ and $\{r_N\}$. $\{l_N\}$ and $\{r_N\}$ are used to define that the sequence $|\lambda_{i,1}|, \ldots, |\lambda_{N,1}|$ can be divided into “asymptotically independent” groups $(\xi_{(i-1)r_N+1}, \ldots, \xi_{ir_N})$ of size $r_N$ each. A special case of Lemma 1 is that $|\lambda_{i,1}|$ is independent, then $\pi_N(1; \tau) \to 1$ and
$G_{1,m}(\tau) = \lim_{N \to \infty} P(|\lambda_{(m)}| \leq u_{1,N}(\tau)) = e^{-\tau} \sum_{t=0}^{m-1} \frac{\rho^t}{t!}$. If the tail of $|\lambda_{i,1}|$ follows an extreme value distribution with parameter $(\mu, \sigma, \xi)$, then $u_{1,N}(\tau) = a_{1,N}(\mu + \sigma \left( \frac{\tau - \xi}{\xi} \right)) + b_{1,N}$ for some normalizing sequences $\{a_{1,N} > 0\}$ and $\{b_{1,N}\}$ and $G_{1,m}(\tau)$ is the same as Theorem 3.4 in [Coles et al. (2001)]. Equipped with Lemma 1, we can provide an asymptotic probabilistic lower bound for $\rho$ when $\lambda_{i,1}$ can be dependent.

B Uniform Consistency of Factor Loadings

Let

$$\hat{\lambda}_i - H \lambda_i = \hat{S}^{-1} \left( \frac{1}{N} \sum_{i=1}^{N} \hat{\lambda}_i E(e_i^T e_i)/T + \frac{1}{N} \sum_{i=1}^{N} \hat{\lambda}_i \zeta_{it} + \frac{1}{N} \sum_{i=1}^{N} \hat{\lambda}_i \eta_{it} + \frac{1}{N} \sum_{i=1}^{N} \hat{\lambda}_i \xi_{it} \right)$$

(21)

where $\hat{S}$ are eigenvalues of $\frac{1}{NT} XX^T$ corresponding to eigenvalues $\hat{\Lambda}$, $H$ is some rotation matrix, $\zeta_{it} = e_i^T e_i/T - E(e_i^T e_i)/T$, $\eta_{it} = \lambda_i^T \sum_{t=1}^{T} f_i e_{it}/T$ and $\xi_{it} = \lambda_i^T \sum_{t=1}^{T} f_i e_{it}/T$.

**Lemma 2.** Under Assumptions 1-4,

1. $\max_{t \leq N} \left\| \frac{1}{NT} \sum_{i=1}^{N} \hat{\lambda}_i E(e_i^T e_i) \right\| = O_p(\sqrt{1/N})$

2. $\max_{t \leq N} \left\| \frac{1}{N} \sum_{i=1}^{N} \hat{\lambda}_i \zeta_{it} \right\| = O_p(N^{1/4}/\sqrt{T})$

3. $\max_{t \leq N} \left\| \frac{1}{N} \sum_{i=1}^{N} \hat{\lambda}_i \eta_{it} \right\| = O_p(N^{1/4}/\sqrt{T})$

4. $\max_{t \leq N} \left\| \frac{1}{N} \sum_{i=1}^{N} \hat{\lambda}_i \xi_{it} \right\| = O_p(N^{1/4}/\sqrt{T})$

**Proof.** 1. By the Cauchy-Schwarz inequality and the fact that $\frac{1}{N} \sum_{i=1}^{N} \left\| \hat{\lambda}_i \right\|^2 = O_p(1)$,

$$\max_{t \leq N} \left\| \frac{1}{NT} \sum_{i=1}^{N} \hat{\lambda}_i E(e_i^T e_i) \right\| \leq \max_{t \leq N} \left( \frac{1}{N} \sum_{i=1}^{N} \left\| \hat{\lambda}_i \right\|^2 \frac{1}{N} \sum_{i=1}^{N} (E(e_i^T e_i)/T)^2 \right)^{1/2}$$

$$\leq O_p(1) \max_{t \leq N} \left( \frac{1}{N} \sum_{i=1}^{N} (E(e_i^T e_i)/T)^2 \right)^{1/2}$$

$$\leq O_p(1) \max_{t \leq N} \sqrt{|E(e_i^T e_i)/T|} \max_{t \leq N} \left( \frac{1}{N} \sum_{i=1}^{N} |E(e_i^T e_i)/T| \right)^{1/2}$$

$$= O_p(\sqrt{1/N})$$

by Assumption 3.2.
2. By Cauchy-Schwarz inequality,
\[
\max_{l \leq N} \left\| \frac{1}{N} \sum_{i=1}^{N} \hat{\lambda}_i \zeta_{il} \right\| \leq \max_{l \leq N} \left( \frac{1}{N} \sum_{i=1}^{N} \left\| \hat{\lambda}_i \right\|^2 \frac{1}{N} \sum_{i=1}^{N} \zeta_{il}^2 \right)^{1/2} \leq O_p(1) \max_{l \leq N} \left( \frac{1}{N} \sum_{i=1}^{N} \zeta_{il}^2 \right)^{1/2} = O_p(N^{1/4}/\sqrt{T}).
\]

It follows from Assumption 3.4 that \(E(\frac{1}{N} \sum_{i=1}^{N} \zeta_{il}^2)^2 \leq \max_{i,l} E \zeta_{il}^4 = O(T^{-2})\). From Markov inequality and Boole’s inequality (the union bound), \(\max_{l \leq N} \frac{1}{N} \sum_{i=1}^{N} \zeta_{il}^2 = O_p(N^{1/4}/\sqrt{T})\).\footnote{Denote \(y_t = \frac{1}{N} \sum_{i=1}^{N} \zeta_{it}^2 \). \(\exists M_1, Ey_t^2 \leq M_1/T^2\). We have \(\forall \epsilon, P(\max_{l \leq N} y_t > NM_1/(T^2 \epsilon)) = P(\exists i, \, y_t > NM_1/(T^2 \epsilon)) \leq \sum_{i=1}^{N} P(y_t > NM_1/(T^2 \epsilon)) \leq \sum_{i=1}^{N} \frac{Ey_t^2}{NM_1/(T^2 \epsilon)} \leq \epsilon\) by Markov Inequality and the union bound. Thus, \(\max_{l \leq N} y_t = O_p(N^{1/4}/\sqrt{T})\).}

3. \(E \left\| \frac{1}{\sqrt{T}} \sum_{t=1}^{T} f_t e_{it} \right\|^4 \leq M\) by Assumption 4. Using Markov inequality and Boole’s inequality (the union bound), yield \(\max_{l \leq N} \left\| \frac{1}{T} \sum_{t=1}^{T} f_t e_{it} \right\| = O_p(N^{1/4}/\sqrt{T})\). Thus,
\[
\max_{l \leq N} \left\| \frac{1}{N} \sum_{i=1}^{N} \hat{\lambda}_i \eta_{il} \right\| \leq \left\| \frac{1}{N} \sum_{i=1}^{N} \hat{\lambda}_i \hat{\lambda}_i^\top \right\| \max_{l \leq N} \left\| \frac{1}{T} \sum_{t=1}^{T} f_t e_{it} \right\| = O_p(N^{1/4}/\sqrt{T})
\]
follows from \(\left\| \frac{1}{N} \sum_{i=1}^{N} \hat{\lambda}_i \hat{\lambda}_i^\top \right\| \leq \left( \frac{1}{N} \sum_{i=1}^{N} \left\| \hat{\lambda}_i \right\|^2 \right)^{1/2} \left( \frac{1}{N} \sum_{i=1}^{N} \left\| \hat{\lambda}_i \right\|^2 \right)^{1/2} = O_p(1)\) by Assumption 2.

4. By Assumption 4 \(\left\| \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} f_t e_{it} \hat{\lambda}_i \right\| \leq \left( \frac{1}{N} \sum_{i=1}^{N} \left\| \hat{\lambda}_i \right\|^2 \max_{l \leq N} \left\| \frac{1}{T} \sum_{t=1}^{T} f_t e_{it} \right\|^2 \right)^{1/2} = O_p(\sqrt{1/T})\). In addition, since \(E \left\| \lambda_i \right\|^4 < M\) from Assumption 2, \(\max_{l \leq N} \left\| \lambda_i \right\| = O_p(N^{1/4})\).\footnote{\(\forall \epsilon, P(\max_{l \leq N} \left\| \lambda_i \right\|^4 > NM/\epsilon) \leq \sum_{i=1}^{N} P\left(\left\| \lambda_i \right\|^4 > NM/\epsilon\right) = \sum_{i=1}^{N} \frac{E \left\| \lambda_i \right\|^4}{NM/\epsilon} \leq \epsilon\) by Markov Inequality and the union bound. Thus, \(\max_{l \leq N} \left\| \lambda_i \right\|^4 = O_p(N)\) and \(\max_{l \leq N} \left\| \lambda_i \right\| = O_p(N^{1/4})\).}

Thus,
\[
\max_{l \leq N} \left\| \frac{1}{N} \sum_{i=1}^{N} \hat{\lambda}_i \xi_{il} \right\| \leq \max_{l \leq N} \left\| \lambda_i \right\| \left\| \frac{1}{N} \sum_{i=1}^{N} \sum_{t=1}^{T} f_t e_{it} \hat{\lambda}_i \right\| = O_p(N^{1/4}/\sqrt{T}).
\]

\[\square\]

**Proof of Theorem 4**

\[\hat{\lambda}_i - H \lambda_i = \hat{S}^{-1} \left( \frac{1}{N} \sum_{i=1}^{N} \hat{\lambda}_i E(e_i^\top e_i)/T + \frac{1}{N} \sum_{i=1}^{N} \hat{\lambda}_i \zeta_{il} + \frac{1}{N} \sum_{i=1}^{N} \hat{\lambda}_i \eta_{il} + \frac{1}{N} \sum_{i=1}^{N} \hat{\lambda}_i \xi_{il} \right).\]

Under Assumptions 1.4 from Lemma A.3 in Bai (2003), we have \(\hat{S} \overset{P}{\to} S\), where \(S = diag(s_1, \ldots, s_K)\) are the eigenvalues of \(\Sigma_F^{1/2} \Sigma A \Sigma_F^{1/2}\). Since from Assumptions 1 and 2, eigenvalues of \(\Sigma_F \Sigma_A\) are bounded away from both zero and infinity. Thus, diagonal elements in...
\( \hat{S} \) are bounded away from zero and infinity with probability 1, therefore, diagonal elements in \( \hat{S}^{-1} \) are bounded away from zero and infinity with probability 1.

\[
\max_{l \leq N} \left\| \hat{\lambda}_l - H \lambda_l \right\| \leq \left\| \hat{S}^{-1} \right\| \left\{ \max_{l \leq N} \left\| \frac{1}{NT} \sum_{i=1}^{N} \hat{\lambda}_l E(e_i^T e_l) \right\| + \max_{l \leq N} \left\| \frac{1}{N} \sum_{i=1}^{N} \hat{\lambda}_l \eta_{il} \right\| \right\}
\]

\[
= O_p(1)(O_p(\sqrt{1/N}) + O_p(N^{1/4}/\sqrt{T})) = O_p(\sqrt{1/N} + N^{1/4}/\sqrt{T})
\]

from Lemma 2

\[\square\]

### C Generalized Correlation Between True Factors and Proximate Factors

**Lemma 3.** Under Assumptions 1-4, \( \forall \epsilon, \delta, \exists N_0, T_0, \) when \( N > N_0, T > T_0, \) with probability \( 1 - \delta, \) if \( \left| (\Lambda H)_{(m),j} \right| \geq c + 2\epsilon, \)

\[
\left| (\Lambda H)_{j_i,j} \right| > c,
\]

where \( \left| (\Lambda H)_{(m),j} \right| \) is the \( m \)-th order statistic of \( \left| (\Lambda H)_{j} \right| \) and \( j_i \) is the index of \( i \)-th order statistic of the \( j \)-th estimated loading \( |\hat{\lambda}_j| \).

**Proof.** Under Assumptions 1-4 from Theorem 1,

\[
\max_{l \leq N} \left\| \hat{\lambda}_l - H^T \lambda_l \right\| = O_p(1/\sqrt{N} + N^{1/4}/\sqrt{T})
\]

Thus, \( \forall j \leq K \max_{l \leq N} |\hat{\lambda}_l - H^T \lambda_l| = O_p(1/\sqrt{N} + N^{1/4}/\sqrt{T}) \). In other words, \( \forall \epsilon, \delta, \) there exist \( N_0 \) and \( T_0 \), such that \( \forall N > N_0, T > T_0, \)

\[
P(\max_{i} \left| \hat{\lambda}_{i,j} - (\Lambda H)_{i,j} \right| > \epsilon) < \delta,
\]

therefore, we have \( \forall N > N_0, T > T_0, \)

\[
P(\max_{i} \left| \hat{\lambda}_{i,j} - (\Lambda H)_{i,j} \right| > \epsilon) < \delta
\]

follows from \( \left| \hat{\lambda}_{i,j} \right| - \left| (\Lambda H)_{i,j} \right| \leq \left| \hat{\lambda}_{i,j} - (\Lambda H)_{i,j} \right| \). We have with probability at least \( 1 - \delta, \forall 1 \leq i \leq m, \) \( |(\Lambda H)_{j_i,j} - \epsilon < |\tilde{\lambda}_{j_i,j} < |(\Lambda H)_{j_i,j} | + \epsilon \) and \( |(\Lambda H)_{(m),j} - \epsilon < |\tilde{\lambda}_{(m),j} | < |(\Lambda H)_{(m),j} | + \epsilon \).

From \( \forall 1 \leq i \leq m, \) \( |\tilde{\lambda}_{j_i,j} | \geq \min(|\hat{\lambda}_{j_i,1}|, \cdots, |\hat{\lambda}_{j_i,m}|) \), with probability \( \geq 1 - \delta, \)

\[
|\Lambda H)_{j_i,j} | + \epsilon > |\tilde{\lambda}_{j_i,j} | \geq \min(|\hat{\lambda}_{j_i,1}|, \cdots, |\hat{\lambda}_{j_i,m}|) > \min(|(\Lambda H)_{(1),j} | - \epsilon, \cdots, |(\Lambda H)_{(m),j} | - \epsilon) = |(\Lambda H)_{(m),j} | - \epsilon
\]

Therefore, if \( |(\Lambda H)_{(m),j} | - 2\epsilon \geq c, \) with probability \( \geq 1 - \delta, \forall 1 \leq i \leq m, |(\Lambda H)_{j_i,j} | > c. \) \[\square\]
Lemma 4. Under Assumptions \[2\] and \[4\] as \(N, T \to \infty\),

\[
\rho = \text{tr} \left( I + \left( \frac{F^T F}{T} \right)^{-1/2} \left( \frac{W^T \Lambda}{T} \right)^{-1} \frac{W^T e e^T \tilde{W}}{T} \left( \frac{F^T F}{T} \right)^{-1/2} \right)^{-1} + o_p(1)
\]

Proof. Denote \(Q = \Lambda^T \tilde{W}\). Given \(\tilde{F} = X^T \tilde{W} (\tilde{W}^T \tilde{W})^{-1}\) and \(X = \Lambda F^T + e\), we have

\[
F^T \tilde{F}/T = (F^T F/T)(\Lambda^T \tilde{W})(\tilde{W}^T \tilde{W})^{-1} + (F^T e e^T \tilde{W}/T)(\tilde{W}^T \tilde{W})^{-1} = (F^T F/T)Q(\tilde{W}^T \tilde{W})^{-1} + o_p(1)
\]

follows from \(\forall i, e_i F/T = o_p(1)\) from Assumption \[4\] and \(\forall i, j, \tilde{w}_{i,j}\) to be bounded because \(\tilde{W}^T \tilde{W}_j = 1\) and each column in \(\tilde{W}\) has \(m\) (fixed) nonzero entries, so \(F^T e e^T \tilde{W}/T = o_p(1)\).

\(\tilde{F}^T \tilde{F}/T\) has

\[
\tilde{F}^T \tilde{F}/T = (\tilde{W}^T \tilde{W})^{-1} (F \Lambda^T \tilde{W} + e e^T \tilde{W})^T (F \Lambda^T \tilde{W} + e e^T \tilde{W})(\tilde{W}^T \tilde{W})^{-1}/T
\]

\[
= (\tilde{W}^T \tilde{W})^{-1} \left( Q^T (F^T F/T)Q + (\tilde{W}^T e F/T)Q + Q^T (F^T e e^T \tilde{W}/T) + \tilde{W}^T ee^T \tilde{W}/T \right) (\tilde{W}^T \tilde{W})^{-1}
\]

follows from \(Q = [q_{ij}] = \Lambda^T \tilde{W}\), where \(q_{ij} = \sum_{i=1}^m \lambda_{i,j} \tilde{W}_{i,j}\). \(\lambda_{i,j} = O_p(1)\) from assumption \[2\] and then \(q_{ij} = O_p(1)\).

Plug the above into \(\rho\) and

\[
\rho = \text{tr} \left( (F^T F/T)^{-1} (F^T \tilde{F}/T)(\tilde{F}^T \tilde{F}/T)^{-1} (F^T F/T) \right)
\]

\[
= \text{tr} \left( \left( \frac{F^T F}{T} \right)^{-1} \left( \frac{F^T F Q}{T} \right) \left( \frac{Q^T (F^T F + (Q^T)^{-1} \tilde{W}^T e e^T \tilde{W} Q^{-1}) Q}{T} \right) \left( \frac{F^T F}{T} \right)^{-1} \right) + o_p(1)
\]

\[
= \text{tr} \left( \left( \frac{F^T F}{T} \right)^{-1/2} \left( \frac{F^T F + (Q^T)^{-1} \tilde{W}^T e e^T \tilde{W} Q^{-1}}{T} \right)^{-1} \left( \frac{F^T F}{T} \right)^{1/2} \right) + o_p(1)
\]

\[
= \text{tr} \left( \left( I + \left( \frac{F^T F}{T} \right)^{-1/2} \left( (Q^T)^{-1} \tilde{W}^T e e^T \tilde{W} Q^{-1} \right)^{-1} \left( \frac{F^T F}{T} \right)^{-1/2} \right)^{-1} \right) + o_p(1)
\]

\[\square\]

Lemma 5. If indices of nonzeros entries in columns of \(\tilde{W}\) do not overlap, then

\[
\frac{1}{T} \tilde{W}^T e e^T \tilde{W} \leq (1 + h(m)) o_e^2 I_K + o_p(1)
\]
Proof. Let the indices of nonzero entries in $\widehat{W}_j$ be $j_1, \ldots, j_m$. From Cauchy-Schwartz inequality, the $(j, j)$ element in $\frac{1}{T} \widehat{W}_j^\top ee^\top \widehat{W}_j$ has

$$\frac{1}{T} \widehat{W}_j^\top ee^\top \widehat{W}_j = \frac{1}{T} \sum_{i=1}^{m} \sum_{k=1}^{m} \bar{w}_{i,j} \bar{w}_{j,k} e_i^\top e_k$$

$$= \frac{1}{T} \sum_{i=1}^{m} \bar{w}_{i,j}^2 e_i^\top e_i + \frac{1}{T} \sum_{i \neq k} \bar{w}_{i,j} \bar{w}_{j,k} e_i^\top e_k$$

$$\leq \sigma_e^2 \sum_{i=1}^{m} \bar{w}_{i,j}^2 + \left( \sum_{i \neq k} \bar{w}_{i,j}^2 \bar{w}_{j,k}^2 \right)^{1/2} \left( \sum_{i \neq k} \left( \frac{1}{T} e_i^\top e_k \right)^2 \right)^{1/2} + o_p(1) \quad (22)$$

$$\leq \sigma_e^2 + \left( \sum_{i \neq k} \bar{w}_{i,j}^2 \bar{w}_{j,k}^2 \right)^{1/2} \left( \sum_{i \neq k} \tau_{j,k}^2 \right)^{1/2} + o_p(1) \quad (23)$$

$$\leq (1 + h(m)) \sigma_e^2 + o_p(1) \quad (24)$$

where $e_i \in \mathbb{R}^{T \times 1}$ is the $i$-th row of $e$. Inequality (22) holds from the definition of $\sigma_e^2$ in Assumption 3. Inequality (23) holds from the stationarity of $e_i$ in Assumption 3.2, $\frac{1}{T} e_i^\top e_i = \tau_{j,k} + o_p(1) \leq \sigma_e^2 + o_p(1)$. Inequality (24) holds from $\sum_{i \neq k} \bar{w}_{i,j}^2 \bar{w}_{j,k}^2 \leq \left( \sum_{i=1}^{m} \bar{w}_{i,j}^2 \right)^2 = 1$ and the definition of $h(m)$, $\sum_{i \neq k} \tau_{j,k}^2 \leq (h(m))^2 \sigma_e^2$.

The $(j, k)$ element in $\frac{1}{T} \widehat{W}_j^\top ee^\top \widehat{W}_j$ is 0 given there are no overlapping nonzero elements among loadings. Thus, we have

$$\frac{1}{T} \widehat{W}_j^\top ee^\top \widehat{W}_j \leq (1 + h(m)) \sigma_e^2 I_K + o_p(1).$$

\[ \square \]

Lemma 6. Under Assumptions 1 and assume $K = K_0$. Let $H = \frac{F^T F}{N} \hat{\Lambda} (\hat{S})^{-1}$, then $H$ is invertible. Let $\hat{S} = H^{-1} \frac{F^T F}{N} (H^\top)^{-1}$, we have

$$\hat{S} = \hat{S} + o_p(1)$$

Proof. Under Assumptions 1 and Theorem 1 in Bai and Ng (2002), $\text{rank} \left( \frac{F^T F}{N} \hat{\Lambda} \hat{\Lambda}^\top \right) = \min(K_0, K)$. Since $K = K_0$, $\frac{F^T F}{N} \hat{\Lambda} \hat{\Lambda}^\top$ is invertible. $\hat{S}$ is invertible by Lemma A.3 in Bai (2003) and Assumptions 1. Substituting $X = \Lambda F^\top + e$ into $\frac{1}{N} \hat{\Lambda}^\top \left( \frac{1}{N} X X^\top \right) \hat{\Lambda} = \hat{S}$, from Equation (21) and Theorem 1 in Bai (2003), we have

$$\hat{S} = \frac{\hat{\Lambda} \hat{\Lambda}^\top}{N} \frac{F^T F}{T} \frac{\Lambda \Lambda^\top}{N} + o_p(1).$$

Plug it into $H = \frac{F^T F}{N} \hat{\Lambda} \hat{\Lambda}^\top (\hat{S})^{-1}$,

$$H = \left( \frac{\hat{\Lambda} \hat{\Lambda}^\top}{N} \right)^{-1} + o_p(1).$$
Thus,

\[ \tilde{S} = H^{-1}F^T F \left( H^T \right)^{-1} = \left( \left( \frac{\hat{\Lambda}^T \Lambda}{N} \right)^{-1} \right)^{-1} F^T F \left( \left( \frac{\hat{\Lambda}^T \Lambda}{N} \right)^{-1} \right)_T^{-1} + o_p(1) = \hat{S} + o_p(1) \]

\[ \] 

**Lemma 7.** Under Assumption \[ \square \] let \( S = \text{diag}(s_1, s_2, \ldots, s_K) \) be the diagonal matrix consisting the eigenvalues of \( \Sigma_F \Sigma_A \) in decreasing order, \( H = \frac{F^T F \Lambda^T \Lambda}{N} \tilde{(S)}^{-1} \) and \( U = \Lambda H S^{1/2} \), where \( S^{1/2} = \text{diag}(s_1^{1/2}, s_2^{1/2}, \ldots, s_K^{1/2}) \), \( \forall i, l \)

\[ \frac{1}{T} \Lambda_i^T F^T F \Lambda_i = U_i^T U_l + o_p(1). \]

Furthermore, rescale each column of \( U \odot M \) to get \( \tilde{U} \),

\[ \tilde{U} = \begin{bmatrix} U_1 \odot M_1 & U_2 \odot M_2 & \cdots & U_K \odot M_K \end{bmatrix} \]

where \( U_j \) and \( M_j \) are \( j \)-th column in \( U \) and \( M \) is the mask matrix defined in \( \square \). We have

\[ \frac{1}{T} \tilde{W}^T (\Lambda F^T F \Lambda) \tilde{W} = \tilde{U}^T U^T \tilde{U} + o_p(1) \]

**Proof.** \( \tilde{S} \) is defined as \( \tilde{S} = H^{-1} \frac{F^T F}{T} \left( H^T \right)^{-1} \), so \( \frac{1}{T} \Lambda F^T F \Lambda^T = \Lambda H \tilde{S} H^T \Lambda^T \). Under Assumptions \( \square \) and from Lemma \( \square \) \( \tilde{S} = \tilde{S} + o_p(1) \). From Lemma A.3 in \[ Bai (2003) \], \( \tilde{S} = \frac{1}{N} \tilde{\Lambda}^T \left( \frac{X^T X}{N} \right) \tilde{\Lambda} = S + o_p(1) \). Thus, \( S = \tilde{S} + o_p(1) \). Since \( U = \Lambda H S^{1/2}; \)

\[ \frac{1}{T} \lambda_i^T F^T F \lambda_i = \lambda_i^T H \tilde{S} H^T \lambda_i = \lambda_i^T H S H^T \lambda_i + o_p(1) = U_i^T u_l + o_p(1), \]

where \( \lambda_i \in R^{K \times 1} \) and \( u_i \in R^{K \times 1} \) are the transposes of \( i \)-th rows in \( \Lambda \) and \( U \). From Theorem \( \square \) \( \lambda_i = H^T \lambda_i + o_p(1) \), together with \( \tilde{S} = S + o_p(1) \), we have \( \tilde{S}^{1/2} \hat{\lambda}_i = u_i + o_p(1) \). Moreover, \( \forall j \), \( \frac{\lambda_j s_j^{1/2} \odot M_j}{\| U_j \odot M_j \|} = \frac{\hat{\lambda}_j \odot M_j}{\| \hat{\lambda}_j \odot M_j \|} \). Since \( \tilde{W} = \begin{bmatrix} \frac{\hat{\lambda}_1 \odot M_1}{\| \hat{\lambda}_1 \odot M_1 \|} & \frac{\hat{\lambda}_2 \odot M_2}{\| \hat{\lambda}_2 \odot M_2 \|} & \cdots & \frac{\hat{\lambda}_K \odot M_K}{\| \hat{\lambda}_K \odot M_K \|} \end{bmatrix} \) and \( \tilde{U} = \begin{bmatrix} U_1 \odot M_1 & U_2 \odot M_2 & \cdots & U_K \odot M_K \end{bmatrix} \), we have for each cross-section unit \( i \),

\[ \tilde{u}_i = \hat{u}_i + o_p(1) \]

and therefore

\[ \frac{1}{T} \tilde{W}^T (\Lambda F^T F \Lambda^T) \tilde{W} = \tilde{W}^T (U U^T) \tilde{W} + o_p(1) = \tilde{U}^T U U^T \tilde{U} + o_p(1) \]

\[ \square \]
Proof of Proposition 4. $Q$ is defined in Lemma 4 as $Q = \Lambda^T \tilde{W}$. Thus, $Q^T \left( \frac{F^T F}{T} \right) Q = \tilde{W}^T (UU^T) \tilde{W}$. From Lemmas 4 and 5, we have

$$
\rho \geq \text{tr} \left( \left( I + \left( \frac{F^T F}{T} \right)^{-1/2} \left( Q^T \right)^{-1} \left( (1 + h(m))\sigma^2_e \right) Q^{-1} \left( \frac{F^T F}{T} \right)^{-1/2} \right)^{-1} \right) + o_p(1)
$$

$$
= \left( 1 + \frac{(1 + h(m))\sigma^2_e}{\frac{1}{T} \tilde{W}^T (\Lambda F^T F \Lambda^T) \tilde{W}} \right)^{-1} + o_p(1).
$$

Since $\tilde{S} = H^{-1} \frac{F^T F}{T} (H^T)^{-1}$, and in one factor model, $\tilde{S}$ and $H$ are scalers, $\sigma^2_f = \frac{F^T F}{T} + o_p(1) = \tilde{S} H^2 + o_p(1)$, and $\hat{\lambda}_{i1} = \lambda_{i1} H + o_p(1), \forall i$, 

$$
\frac{1}{T} \tilde{W}^T (\Lambda F^T F \Lambda^T) \tilde{W} = (\tilde{W}^T \tilde{\Lambda}) \tilde{S} (\tilde{\Lambda}^T \tilde{W}) + o_p(1)
$$

$$
= \tilde{S} \left( \sum_{i=1}^{m} \hat{\lambda}_{i1}^2 \right) / \left( \sum_{i=1}^{m} \hat{\lambda}_{i1}^2 \right) + o_p(1)
$$

$$
= \sigma^2_f \sum_{i=1}^{m} \lambda_{i1}^2 + o_p(1).
$$

Plug it into the inequality of $\rho$,

$$
\rho \geq \left( 1 + \frac{(1 + h(m))\sigma^2_e}{\sigma^2_f \sum_{i=1}^{m} \lambda_{i1}^2} \right)^{-1} + o_p(1).
$$

If $\forall y, F_{|\lambda_{i1}|}(y) = P(|\lambda_{i1}| \leq y)$ is continuous in $y$ and from the fact that convergence in probability implies convergence distribution,

$$
\lim_{N,T \to \infty} P(\rho > \rho_0) \geq \lim_{N \to \infty} P \left( \left( 1 + \frac{(1 + h(m))\sigma^2_e}{\sigma^2_f \sum_{i=1}^{m} \lambda_{i1}^2} \right)^{-1} > \rho_0 \right).
$$

Moreover,

$$
\left( 1 + \frac{(1 + h(m))\sigma^2_e}{\sigma^2_f \sum_{i=1}^{m} \lambda_{i1}^2} \right)^{-1} > \rho_0 \iff \frac{\sigma^2_f}{\sigma^2_e} \sum_{i=1}^{m} \lambda_{i1}^2 > \frac{\rho_0}{1 - \rho_0}.
$$

Let the order statistic of the absolute value of $\lambda_{i1}$ be $|\lambda_{(1),1}| \geq |\lambda_{(2),1}| \geq \cdots \geq |\lambda_{(N),1}|$. From Lemma 3, we can ignore term $H$ since it is a scaler, and we know if $|\lambda_{(m),1}| > c$, let $\epsilon = \frac{1}{T} (|\lambda_{(m),1}| - c)$, then $|\lambda_{(m),1}| > c + 2\epsilon$ and $\forall 0 < \delta < 1, \exists N_0, T_0$, such that $N > N_0$, $T > T_0$, with probability at least $1 - \delta$, $|\lambda_{11}| > c$. Thus, as $N, T \to \infty$, $\forall c, \forall 1 \leq i \leq m$, if

$^{34}F_{|\lambda_{i1}|}(y)$ is continuous in $y$ implies $\lim_{N \to \infty} P \left( \left( 1 + \frac{(1 + h(m))\sigma^2_e}{\sigma^2_f \sum_{i=1}^{m} \lambda_{i1}^2} \right)^{-1} > \rho_0 \right)$ is continuous in $\rho_0$.
$\lambda_{(m),1} > c$, then $\lambda_{1,1} > c$ with probability 1. Denote $y_m = \sqrt{\frac{1 + h(m)}{m} \frac{\sigma^2}{\sigma_{f_1}^2} \frac{\rho_0}{1 - \rho_0}}$. As $N, T \to \infty$, with probability 1,

$$|\lambda_{(m),1}| > y_m \Rightarrow \frac{\sigma_f^2}{(1 + h(m))\sigma_e^2} \sum_{i=1}^{m} \lambda^2_{(i),1} > \frac{\rho_0}{1 - \rho_0} \Rightarrow \frac{\sigma_f^2}{\sum_{i=1}^{m} \lambda^2_{1,i,1}} > \frac{\rho_0}{1 - \rho_0}.$$ Thus,

$$\lim_{N,T\to\infty} P(\rho > \rho_0) \geq \lim_{N\to\infty} P \left( \left( \frac{1 + h(m)}{\sigma_{f_1}^2} \sum_{i=1}^{m} \lambda^2_{1,i,1} \right)^{-1} > \rho_0 \right) \geq \lim_{N\to\infty} P(|\lambda_{(m),1}| > y_m).$$

Since

$$P(|\lambda_{(m),1}| > y_m) = 1 - P(|\lambda_{(1),1}| \leq y_m) - \sum_{j=1}^{m} P \left( |\lambda_{(j),1}| \geq y_m, |\lambda_{(j+1),1}| \leq y_m \right)$$

and $F_{|\lambda_{1,1}|}(y)$ is continuous in all $y$, we have for a specific $m$ and a given level $\rho_0$, as $N, T \to \infty$,

$$\lim_{N,T\to\infty} P(\rho > \rho_0) \geq 1 - \lim_{N\to\infty} \sum_{j=0}^{m-1} \left( \begin{array}{c} N \\ j \end{array} \right) (1 - F_{|\lambda_{1,1}|}(y_m))^j F_{|\lambda_{1,1}|}(y_m)^{N-j}.$$ 

Proof of Proposition For a particular $0 < \rho_0 < 1$ and $m$, if $F_{|\lambda_{1,1}|}(y_m) < 1$, $\forall 0 \leq j \leq m$, $\lim_{N\to\infty} \left( \begin{array}{c} N \\ j \end{array} \right) (1 - F_{|\lambda_{1,1}|}(y_m))^j F_{|\lambda_{1,1}|}(y_m)^{N-j} \to 0$. Thus, as $N \to \infty$,

$$\lim_{N,T\to\infty} \sum_{j=0}^{m-1} \left( \begin{array}{c} N \\ j \end{array} \right) (1 - F_{|\lambda_{1,1}|}(y_m))^j F_{|\lambda_{1,1}|}(y_m)^{N-j} \to 0.$$ Since $P(\rho > \rho_0) \leq 1$, we have as $N, T \to \infty$,

$$P(\rho > \rho_0) \to 1.$$ 

Proof of Theorem Denote $\rho_0 = \frac{m \sigma^2 \sigma_{u,N}(\tau)}{(1 + h(m)) \sigma_e^2 + m \sigma^2 \sigma_{u,N}(\tau)}$. Then $y_m = \sqrt{\frac{(1 + h(m)) \sigma^2}{m} \frac{\rho_0}{1 - \rho_0}} = u_{1,N}(\tau)$. From the proof of Proposition,

$$\lim_{N,T\to\infty} P(\rho > \rho_0) \geq \lim_{N\to\infty} P \left( \left( \frac{1 + h(m)}{\sigma_{f_1}^2} \sum_{i=1}^{m} \lambda^2_{1,i,1} \right)^{-1} > \rho_0 \right) \geq \lim_{N\to\infty} P(|\lambda_{(m),1}| > y_m),$$
Since $G_{1,m}(\tau)$ is continuous in $\tau$ by the definition of $G_{1,m}(\tau)$,
\[
\lim_{N,T \to \infty} P \left( \rho > \frac{m\sigma^2_{f_1} u_{1,N}^2(\tau)}{(1 + h(m))\sigma^2_e + m\sigma^2_{f_1} u_{1,N}^2(\tau)} \right) \geq \lim_{N \to \infty} P(|\lambda_{(m),1}| > u_{1,N}(\tau)) \\
= 1 - G_{1,m}(\tau),
\]
where $G_{1,m}(\tau)$ is defined in Lemma 1.

**Proof of Theorem 4** For any small symmetric matrix $M$, from Taylor expansion and mean value theorem, there exist a symmetric matrix $\tilde{M}$,
\[
tr((I + M)^{-1}) = tr(I - M + \frac{1}{2}\tilde{M}^2) \geq tr(I - M),
\]
therefore,
\[
\rho = tr \left( \left( I + \left( \frac{F^T F}{T} \right)^{-1/2} (Q^T)^{-1} \frac{\tilde{W}^T e e^T \tilde{W}}{T} Q^{-1} \left( \frac{F^T F}{T} \right)^{-1/2} \right)^{-1} \right) + o_p(1) \tag{25}
\]
\[
> tr \left( I - \left( \frac{F^T F}{T} \right)^{-1/2} (Q^T)^{-1} \frac{\tilde{W}^T e e^T \tilde{W}}{T} Q^{-1} \left( \frac{F^T F}{T} \right)^{-1/2} \right) + o_p(1)
\]
\[
= K - tr \left( \frac{\tilde{W}^T e e^T \tilde{W}}{T} \left( Q^{-1} \left( \frac{F^T F}{T} \right)^{-1} \right)^{-1} \right) + o_p(1)
\]
\[
= K - tr \left( \frac{\tilde{W}^T e e^T \tilde{W}}{T} \left( \frac{1}{T} \tilde{W}^T (\Lambda F^T F \Lambda^T) \tilde{W} \right)^{-1} \right) + o_p(1)
\]
\[
\geq K - (1 + h(m))\sigma^2_e tr \left( \frac{1}{T} \tilde{W}^T (\Lambda F^T F \Lambda^T) \tilde{W} \right)^{-1} + o_p(1) \tag{26}
\]
\[
= K - (1 + h(m))\sigma^2_e tr(A^{-1}) + o_p(1) \tag{27}
\]
where $A = [a_{ij}] = \tilde{U}^T U U^T \tilde{U}$. Equation (25) follows from Lemma 4. Inequality (26) follows from Lemma 5. Equation (27) follows from Lemma 7. If $F_{[v_1]}(v) = F_{[v_1, \ldots, v_{K}]}(v)$ is continuous in $v$, and from $U = VS^{1/2}$, then $F_{[u]}(u)$ is continuous in $u$. From $\rho = K - (1 + h(m))\sigma^2_e tr(A^{-1}) + o_p(1)$ and the fact that convergence in probability implies convergence in distribution,
\[
\lim_{N,T \to \infty} P(\rho > \rho_0) \geq \lim_{N \to \infty} P(K - (1 + h(m))\sigma^2_e tr(A^{-1}) > \rho_0) = \lim_{N \to \infty} P \left( tr(A^{-1}) < \frac{K - \rho_0}{(1 + h(m))\sigma^2_e} \right)
\]
Indices of nonzero entries $\{j_1, j_2, \ldots, j_m\}$ in $\tilde{W}_j$ are selected to be the indices of largest $m$ entries in $|\Lambda_j|$, so indices of nonzero entries in $U_j$ are $\{j_1, j_2, \ldots, j_m\}$ as well. Each entry in $A$ can be written as
\[
a_{jj} = \frac{\sum_{k=1}^{K} (\sum_{i=1}^{m} u_{j_i,j} u_{j_i,k})^2}{\sum_{i=1}^{m} u_{j_i,j}^2}
\]
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and \( \forall j \neq l \)
\[
a_{jl} = \frac{\sum_{k=1}^{K} (\sum_{i=1}^{m} u_{j,i} u_{j,k}) (\sum_{i=1}^{m} u_{i,k} u_{i,l})}{(\sum_{i=1}^{m} u_{j,i}^2)^{1/2} (\sum_{i=1}^{m} u_{i,l}^2)^{1/2}},
\]
where \( U = VS^{1/2} \). Denote \( A = (U^T U)(U^T \tilde{U}) = DB^T BD \), where \( D \) is a diagonal matrix with \( d_{jj} = (\sum_{i=1}^{m} u_{j,i}^2)^{1/2} \) and the diagonal entries in \( B \) are 1, while the off-diagonal entries have
\[
b_{kl} = \frac{\sum_{i=1}^{m} u_{i,k} u_{i,l}}{\sum_{i=1}^{m} u_{i,l}^2}.
\]

Since \( B^T B \) is positive semidefinite, from the matrix inequality that
\[
\text{tr}(A^{-1}) = \text{tr}((DB^T BD)^{-1}) = \text{tr}(D^{-1}(B^T B)^{-1}D^{-1}) \leq \lambda_{\max}((B^T B)^{-1}) \text{tr}(D^{-2})
\]
\[
= \frac{1}{\lambda_{\min}(B^T B)} \text{tr}(D^{-2}) = \frac{1}{\sigma_{\min}^2(B)} \text{tr}(D^{-2}) = \frac{1}{\sigma_{\min}^2(B)} \sum_{j=1}^{K} \frac{1}{\sum_{i=1}^{m} u_{j,i}^2},
\]

where \( \lambda_{\max}(M) \), \( \lambda_{\min}(M) \), \( \sigma_{\max}(M) \) and \( \sigma_{\min}(M) \) represent the maximum eigenvalue, minimum eigenvalue, maximum singular value and minimum singular value of matrix \( M \). Let the event \( E_1 = \{ \sigma_{\min}(B) \geq \gamma \} \) for some \( \gamma \). From Lemma 3 and the proof of Proposition 1, \( \forall i, j \) \( \forall c \), if \( v_{(m),j} > c, v_{j,j} > c \) with probability 1 as \( N, T \to \infty \). Moreover, \( u_{i,j} = v_{i,j} s_{j,l}^2 \). Similarly, \( \forall i, j \) \( \forall c \), if \( u_{(m),j} > c, u_{j,j} > c \) with probability 1 as \( N, T \to \infty \). Thus, if event \( E_1 \) happens, we have
\[
\sum_{j=1}^{K} \frac{1}{s_j v_{(m),j}^2} < \frac{m(K - \rho_0)^2 \gamma^2}{(1 + h(m)) \sigma_e^2} \Rightarrow \sum_{j=1}^{K} \frac{1}{m s_j v_{(m),j}^2} < \frac{(K - \rho_0)^2 \gamma^2}{(1 + h(m)) \sigma_e^2}
\]
\[
\Rightarrow \sum_{j=1}^{K} \frac{1}{\sum_{i=1}^{m} u_{j,i}^2} < \frac{(K - \rho_0)^2 \gamma^2}{(1 + h(m)) \sigma_e^2} \Rightarrow \frac{1}{\sigma_{\min}^2(B)} \sum_{j=1}^{K} \sum_{i=1}^{m} u_{j,i}^2 < \frac{(K - \rho_0)}{(1 + h(m)) \sigma_e^2}
\]
under \( E_1 \)
\[
\frac{1}{\sigma_{\min}^2(B)} \sum_{j=1}^{K} \sum_{i=1}^{m} u_{j,i}^2 < \frac{K - \rho_0}{(1 + h(m)) \sigma_e^2} \Rightarrow \text{tr}(A^{-1}) < \frac{K - \rho_0}{(1 + h(m)) \sigma_e^2}
\]

Let \( E_2 = \left\{ \sum_{j=1}^{K} \frac{1}{s_j v_{(m),j}^2} < \frac{m(K - \rho_0)^2 \gamma^2}{(1 + h(m)) \sigma_e^2} \right\} \) and \( E_3 = \left\{ \text{tr}(A^{-1}) < \frac{K - \rho_0}{(1 + h(m)) \sigma_e^2} \right\} \). We have
\[
P(E_1 \cap E_3) \geq P(E_1 \cap E_2).
\]
Denote \( \rho_0 = K - \frac{(1 + h(m)) \sigma_e^2}{m^2} \sum_{j=1}^{K} \frac{1}{s_j (a_{j,N_j} z + b_{j,N_j}) \gamma^2} \). We have
\[
\forall j, \ |v_{(m),j}| > u_{j,m}(\tau) \Rightarrow \sum_{j=1}^{K} \frac{1}{s_j v_{(m),j}^2} < \sum_{j=1}^{K} \frac{1}{s_j u_{j,N}^2(\tau)} \Rightarrow \sum_{j=1}^{K} \frac{1}{s_j v_{(m),j}^2} < \frac{m(K - \rho_0)^2 \gamma^2}{(1 + h(m)) \sigma_e^2}
\]
Let \( E_4 = \left\{ \forall j, \ |v_{(m),j}| > u_{j,m}(\tau) \right\} \). We have
\[
P(E_2) \geq P(E_4)
\]
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Since \( P(E_3) \geq P(E_1 \cap E_3) \geq P(E_1 \cap E_2) \geq P(E_2) - P(E_1) \geq P(E_4) - P(E_1^c) \), where \( E_1^c = \{ \sigma_{\min}(B) < \gamma \} \) is the complementary of \( E_1 \). Then we have

\[
\lim_{N,T \to \infty} P(\rho > \rho_0) \geq \lim_{N \to \infty} P \left( \text{tr}(A^{-1}) < \frac{K - \rho_0}{(1 + h(m))\sigma_\varepsilon^2} \right) = \lim_{N \to \infty} P(E_3) \geq \lim_{N \to \infty} (P(E_4) - P(E_1^c))
\]

Since \( G_j^*(z) \) is continuous by definition, we have

\[
\lim_{N,T \to \infty} P \left( \rho \geq K - \frac{(1 + h(m))\sigma_\varepsilon^2}{m\gamma^2} \sum_{j=1}^{K} \frac{1}{\sigma_ju_j^2} \right) \geq \prod_{j=1}^{K} (1 - G_j(m)) - \lim_{N \to \infty} P(\sigma_{\min}(B) < \gamma),
\]

where \( G_j(m) \) is defined in Lemma 1 and Assumption 6.

\[\Box\]

To prove Theorem 5, we first introduce Relative Weyl’s Theorem (Weyl (1912), Corollary 2.3 in Dopico et al. (2000)).

Lemma 8 (Relative Weyl’s Theorem). Let \( A = [a_{ij}] \) be a Hermitian matrix such that \( D = \text{diag}[a_{ii}] \) is invertible and \( \alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n \) its diagonal entries in decreasing order. Let \( \xi_1 \geq \xi_2 \geq \cdots \geq \xi_n \) be the eigenvalues of \( A \). If \( \|D^{-1/2}(A - D)D^{-1/2}\|_2 \leq 1 \), with \( D^{1/2} \) any normal square root of \( D \), then

\[
\frac{\xi_i - \alpha_i}{|\alpha_i|} \leq \|D^{-1/2}(A - D)D^{-1/2}\|_2, \quad i = 1, \cdots, n.
\]

Proof of Theorem 5. Following the proof of Lemma 4, we have

\[
\rho = \text{tr} \left( I + \left( \frac{F^T F}{T} \right)^{1/2} \left( \tilde{W}^P \right)^T \Lambda^{-1} \left( \tilde{W}^P \right)^{1/2} \frac{e e^T}{T} \left( \Lambda^{-1} \tilde{W}^P \right)^{-1} \left( \frac{F^T F}{T} \right)^{1/2} \right) + o_p(1).
\]

Following the proof of Lemma 5, we have

\[
\frac{1}{T} (\tilde{W}^P)^T e e^T \tilde{W}^P \leq (1 + h(m))\sigma_\varepsilon^2 I_K + o_p(1).
\]

Define \( U^P = [U^P_1, U^P_2, \cdots, U^P_K] = V^P = \Lambda H S^{1/2} P \) and

\[
\tilde{U}^P = \begin{bmatrix} U^P_1 \otimes M_1 & U^P_2 \otimes M_2 & \cdots & U^P_K \otimes M_K \end{bmatrix}.
\]

Since \( PP^T = I_K \) and \( P^T P = I_K \), \( UU^T = U^P (U^P)^T = V^P (V^P)^T \). From Lemma 7, we have

\[
\frac{1}{T} \Lambda F^T F \Lambda^T = U^P (U^P)^T + o_p(1).
\]

From Theorem 1, for each cross-section unit \( i \), \( \hat{\lambda}_i = H^T \lambda_i + o_p(1) \), together with \( \hat{S} = S + o_p(1) \), then we have \( \hat{S}^{1/2} \hat{\lambda}_i = u_i + o_p(1) \). Since \( (\hat{w}_i^P)^T = (\hat{\lambda}_i)^T \hat{S}^{1/2} P \) and \( (u_i^P)^T = \lambda_i^T H S^{1/2} P \), we
have \( \tilde{w}_i^P = u_i^P + o_p(1) \). From the definition of \( \tilde{W}^P \) and \( \tilde{U}^P \), we have \( \tilde{w}_i^P = \bar{u}_i^P + o_p(1) \). Define \( Q^P \) as \( Q^P = \Lambda^T \tilde{W}^P \). Similar to the proof of Theorem 4, we have

\[
\rho = tr\left( \left(I + \left( \frac{F^T F}{T} \right)^{-1/2} \left((Q^P)^T - 1 \right) \left(\tilde{W}^P \right)^T \left(Q^P\right)^{-1} \left( \frac{F^T F}{T} \right)^{-1/2} \right)^{-1} \right) + o_p(1)
\]

\[
> K - (1 + h(m))\sigma^2_e tr\left( \left((\tilde{U}^P)^T U^P (U^P)^T \tilde{U}^P \right)^{-1} \right) + o_p(1) \geq K - (1 + h(m))\sigma^2_e tr((A^P)^{-1}) + o_p(1)
\]

where \( A^P = [a^P_{ij}] = (\tilde{U}^P)^T U^P (U^P)^T \tilde{U}^P \).

Let \( j_i \) be the indices satisfying \( |\hat{v}_{i,j}^P| \geq \cdots \geq |\hat{v}_{i,m,j}^P| \) and \( |\hat{v}_{i,j,k}^P / \hat{v}_{i,j}^P| \leq c \) for all \( j, k \). From the definition of \( A^P \),

\[
a^P_{jj} = \frac{\sum_{k=1}^{K} \left( \sum_{i=1}^{m} u_{i,j} u_{i,k}^P \right)^2}{\sum_{i=1}^{m} (u_{i,j})^2}.
\]

Let \( A^P_{S} = [a^P_{Sij}] = D^{-1/2}(A^P - D)D^{-1/2} \),

\[
a^P_{Sjj} = \frac{\sum_{k=1}^{K} \left( \sum_{i=1}^{m} u_{i,j} u_{i,k}^P \right)^2}{\sum_{i=1}^{m} (u_{i,j})^2} \left( \sum_{i=1}^{m} (u_{i,j})^2 \right)^{1/2} \left( \sum_{i=1}^{m} (u_{i,j})^2 \right)^{1/2} = \sum_{k=1}^{K} \left( \sum_{i=1}^{m} u_{i,j} u_{i,k}^P \right)^2 \left( \sum_{i=1}^{m} (u_{i,j})^2 \right)^{1/2} \left( \sum_{i=1}^{m} (u_{i,j})^2 \right)^{1/2}.
\]

Denote \( v_{jj,l}^P = \sum_{i=1}^{m} u_{i,j} u_{i,l}^P \). We have

\[
\left| v_{jj,l}^P \right| \leq v_{jj,j}^P, \forall k \neq j, \left| v_{kl,l}^P \right| \leq v_{kl,l}^P, \forall k \neq l.
\]

Since \( v_{jj,j}^P = u_{j,j}^P, |v_{jj,k}^P / v_{jj,j}^P| < c \) by assumption and Theorem 1, \( |v_{jj,k}^P / v_{jj,j}^P| < c \) by the similar argument as Lemma 3 and the proof of proposition 1

\[
\frac{|v_{jj,j}^P|}{v_{jj,j}^P} = \frac{\sum_{i=1}^{m} u_{i,j} u_{i,j}^P}{\sum_{i=1}^{m} (u_{i,j}^P)^2} < c, \forall j \neq k, \frac{|v_{kl,l}^P|}{v_{kl,l}^P} = \frac{\sum_{i=1}^{m} u_{i,l} u_{i,l}^P}{\sum_{i=1}^{m} (u_{i,l}^P)^2} < c, \forall k \neq l.
\]

We have

\[
a_{Sjj}^P = \frac{\sum_{k=1}^{K} v_{jj,k}^P v_{kl}^P}{(v_{jj,j}^P)^2} \left( \sum_{i=1}^{m} (v_{jj,j}^P)^2 \right)^{1/2} \left( \sum_{i=1}^{m} (v_{kl,l}^P)^2 \right)^{1/2}
\]

\[
\leq \frac{\sum_{k=1}^{K} \left| v_{jj,k}^P \right| |v_{kl,l}^P|}{v_{jj,j}^P (v_{jj,j}^P)^2} \left( \sum_{i=1}^{m} (v_{jj,j}^P)^2 \right)^{1/2} \left( \sum_{i=1}^{m} (v_{kl,l}^P)^2 \right)^{1/2}
\]

\[
\leq \frac{\sum_{k=1}^{K} \left| v_{jj,k}^P \right| |v_{kl,l}^P|}{c (v_{jj,j}^P)^2} \left( \sum_{i=1}^{m} (v_{jj,j}^P)^2 \right)^{1/2} \left( \sum_{i=1}^{m} (v_{kl,l}^P)^2 \right)^{1/2}
\]

\[
< c + c + \sum_{k' \neq j,l} c^2 = c (2 + c(K - 2))
\]
From norm inequality, \( \|D^{-1/2}(A^p - D)D^{-1/2}\|_F \leq \|D^{-1/2}(A^p - D)D^{-1/2}\|_F \),

\[
\|D^{-1/2}(A^p - D)D^{-1/2}\|_F^2 = \sum_{j=1}^{K} \sum_{l \neq j} (a_{p,j,l}^p)^2 < c^2 K(K - 1)(2 + c(K - 2))^2 \triangleq \gamma^2
\]

Let the eigenvalues of \( A^p \) be \( \xi_1, \xi_2, \ldots, \xi_K \) and the order of \( \xi_1, \xi_2, \ldots, \xi_K \) be the same as the order of \( a_{11}, a_{22}, \ldots, a_{KK} \). If \( \gamma \leq 1 \), equivalently, \( 0 < c < 1/(2\sqrt{K(K - 1)}) \) when \( K = 2 \) and \( 0 < c < (\sqrt{1 + (K - 2)/\sqrt{K(K - 1) - 1}})/(K - 2) \) when \( K > 2 \), from Lemma 8,

\[
\left| \frac{\xi_j - a_{jj}^p}{a_{jj}^p} \right| < \gamma, \quad j = 1, 2, \ldots, K
\]

and

\[
\frac{1}{1 + \gamma a_{jj}^p} < \frac{1}{\xi_j} < \frac{1}{1 - \gamma a_{jj}^p}, \quad j = 1, 2, \ldots, K.
\]

Moreover,

\[
a_{jj}^p = \frac{\sum_{k=1}^{K} (\sum_{i=1}^{m} u_{i,j}^p u_{i,k}^p)^2}{\sum_{i=1}^{m} (u_{i,j}^p)^2} \geq \frac{\sum_{i=1}^{m} (u_{i,j}^p)^2}{2} \geq \frac{1}{\xi_j^2}.
\]

\( \bar{v}_{(m),j}^p \) be the \( m \)-th order statistic of the entries in \( |V_j^p| \) that satisfy \( \max_{j, k \neq j} |v_{i,k}^p/v_{i,j}^p| < c \).

\( \forall i, j, \alpha \), if \( \bar{v}_{(m),j}^p = u_{(m),j}^p > \alpha \), then \( u_{i,j}^p > \alpha \) with probability 1 as \( N, T \to \infty \), from \( \hat{u}_i^p = u_i^p + o(1) = u_i^p + o_p(1) \), Lemma 3 and the proof of Proposition 1. Thus, as \( N, T \to \infty \),

\[
\sum_{j=1}^{K} \frac{1}{\bar{v}_{(m),j}^p} < \frac{m(1 - \gamma)(K - \rho_0)}{(1 + h(m))\sigma^2 \varepsilon} \Rightarrow \sum_{j=1}^{K} \frac{1}{\sum_{i=1}^{m} (u_{i,j}^p)^2} < \frac{(1 - \gamma)(K - \rho_0)}{(1 + h(m))\sigma^2 \varepsilon}
\]

\[
\Rightarrow \text{tr}((A^p)^{-1}) < \frac{K - \rho_0}{(1 + h(m))\sigma^2 \varepsilon}
\]

and for a particular \( \rho_0 \),

\[
\lim_{N,T \to \infty} P(\rho > \rho_0) \geq \lim_{N \to \infty} P \left( \text{tr}((A^p)^{-1}) < \frac{K - \rho_0}{(1 + h(m))\sigma^2 \varepsilon} \right)
\]

\[
\geq \lim_{N \to \infty} P \left( \sum_{j=1}^{K} \frac{1}{\bar{v}_{(m),j}^p} < \frac{m(1 - \gamma)(K - \rho_0)}{(1 + h(m))\sigma^2 \varepsilon} \right).
\]

**Proof of Theorem 2** Without loss of generality, we assume \( \tilde{F}^\top \tilde{F}/T = I_r \). Otherwise, we can multiply an invertible rotation matrix \( M \) to \( \tilde{F} \), denoted as \( \hat{F} \), such that \( \tilde{F}^\top \tilde{F}/T = I_r \). Furthermore, \( \rho_{\hat{F},F} = \rho_{\tilde{F},F} \). Let

\[
\hat{\Lambda} = X\hat{F}(\hat{F}^\top \hat{F})^{-1} = X\hat{F}/T.
\]

Note that

\[
\hat{\Lambda} = X\hat{F}/T = X\hat{F}M/T = \bar{\Lambda}(\hat{F}^\top \hat{F})(M/T)^{-1} = \bar{\Lambda} M',
\]

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where $M' = (\hat{F}^T \hat{F})(M/T)^{-1}$ is also an invertible rotation matrix.

The closeness between $\hat{\Lambda}$ and $\Lambda$, $\rho_{\hat{\Lambda}, \Lambda}$, is defined similarly as $\rho_{\hat{\Lambda}, \Lambda}$. Since $M'$ is invertible, $\rho_{\hat{\Lambda}, \Lambda} = \rho_{\hat{\Lambda}, \Lambda}$.

Denote $P$ as $P = F^T \hat{F}/T$. Given $\tilde{\Lambda} = X \hat{F}/T$ and $X = \Lambda F^T + e$, we have
\[
\Lambda^T \tilde{\Lambda}/N = (\Lambda^T \Lambda/(N^2)) + \Lambda e \hat{F}^T/(NT) = (\Lambda^T \Lambda/N)(F^T \hat{F}/T) + o_p(1)
\]
follows from $\forall t$, $\Lambda^T c_t/N = o_p(1)$ from Assumption 4. $\hat{\Lambda}^T \hat{\Lambda}/N$ has
\[
\hat{\Lambda}^T \hat{\Lambda}/N = (X \hat{F}/T)^T (X \hat{F}/T)
\]
\[
= \frac{\hat{F}^T F \Lambda^T \Lambda F^T \hat{F}}{N^2} + \frac{\hat{F}^T e \hat{F}}{NT^2} + o_p(1)
\]

Then we have
\[
\rho_{\hat{\Lambda}, \Lambda} = tr \left( (\Lambda^T \Lambda/N)^{-1} \Lambda^T \hat{\Lambda} \Lambda^T \hat{\Lambda}/N) \right)
\]
\[
= tr \left( \left( \frac{\Lambda^T \Lambda}{N} \right)^{-1} \left( \frac{\Lambda^T F^T \hat{F}}{NT} \right) \left( \frac{\hat{F}^T F \Lambda^T \Lambda F^T \hat{F}}{N^2} + \frac{\hat{F}^T e \hat{F}}{NT^2} \right)^{-1} \left( \frac{\hat{F}^T F \Lambda^T \Lambda}{NT} \right) \right) + o_p(1)
\]
\[
= tr \left( I_K + \left( \frac{\Lambda^T \Lambda}{N} \right)^{-1/2} \left( \frac{\hat{F}^T F}{T} \right)^{-1} \frac{\hat{F}^T e \hat{F}}{NT^2} \left( \frac{\hat{F}^T F}{T} \right)^{-1} \left( \frac{\Lambda^T \Lambda}{N} \right)^{-1/2} \right) + o_p(1)
\]

For any small symmetric matrix $M$, from Taylor expansion and mean value theorem, there exists a symmetric matrix $\tilde{M}$,
\[
tr((I + M)^{-1}) = tr(I - M + \frac{1}{2} \tilde{M}^2) \geq tr(I - M),
\]
we have
\[
\rho_{\hat{\Lambda}, \Lambda} \geq K - tr \left( \left( \frac{\Lambda^T \Lambda}{N} \right)^{-1/2} \left( \frac{\hat{F}^T F}{T} \right)^{-1} \frac{\hat{F}^T e \hat{F}}{NT^2} \left( \frac{\hat{F}^T F}{T} \right)^{-1} \left( \frac{\Lambda^T \Lambda}{N} \right)^{-1/2} \right) + o_p(1)
\]
\[
= K - tr \left( \left( \frac{\hat{F}^T e \hat{F}}{NT^2} \right) \left( \frac{\hat{F}^T F \Lambda^T \Lambda F^T \hat{F}}{NT^2} \right)^{-1} \right) + o_p(1)
\]
\[
\geq K - \sqrt{tr \left( \left( \frac{\hat{F}^T e \hat{F}}{NT^2} \right)^2 \right)} \sqrt{tr \left( \left( \frac{\hat{F}^T F \Lambda^T \Lambda F^T \hat{F}}{NT^2} \right)^{-2} \right)} + o_p(1)
\]
\[
\geq K - tr \left( \frac{\hat{F}^T e \hat{F}}{NT^2} \right) tr \left( \left( \frac{\hat{F}^T F \Lambda^T \Lambda F^T \hat{F}}{NT^2} \right)^{-1} \right) + o_p(1),
\]
followed from \(\text{tr}(M^k) \leq (\text{tr}(M))^k\) for a symmetric positive semidefinite matrix \(M\). For the term \(\text{tr} \left( \left( \frac{\tilde{F}^T F \Lambda \Lambda F^T \tilde{F}}{N T^2} \right)^{-1} \right)\), we have

\[
\text{tr} \left( \left( \frac{\tilde{F}^T F \Lambda \Lambda F^T \tilde{F}}{N T^2} \right)^{-1} \right) = \text{tr} \left( \left( \frac{\Lambda \Lambda F^T F}{N T} \right)^{-1} \left( \frac{(F^T F)^{-1} F^T \tilde{F} \tilde{F}^T F}{T} \right)^{-1} \right)
\]

\[
\leq \sqrt{\text{tr} \left( \left( \frac{\Lambda \Lambda F^T F}{N T} \right)^{-2} \right)} \sqrt{\text{tr} \left( \left( \frac{(F^T F)^{-1} F^T \tilde{F} \tilde{F}^T F}{T^2} \right)^{-2} \right)}
\]

\[
\leq \text{tr} \left( \left( \frac{\Lambda \Lambda F^T F}{N T} \right)^{-1} \right) \text{tr} \left( \left( \frac{(F^T F)^{-1} F^T \tilde{F} \tilde{F}^T F}{T^2} \right)^{-1} \right)
\]

Denote \(A_{\tilde{F}, F} = (F^T F/T)^{-1} (F^T \tilde{F}/T) (\tilde{F}^T \tilde{F}/T)^{-1} (\tilde{F}^T F/T)\) and \(\tilde{A}_{\tilde{F}, F} = (F^T F/T)^{-1} (F^T \tilde{F}/T) (\tilde{F}^T \tilde{F}/T)^{-1} (\tilde{F}^T F/T)\). We have

\[
A_{\tilde{F}, F} = (F^T F/T)^{-1} (F^T \tilde{F} M/T) (M^T \tilde{F}^T \tilde{F} M/T)^{-1} (M^T \tilde{F}^T F/T) = \tilde{A}_{\tilde{F}, F}
\]

Thus,

\[
\text{tr} \left( \left( \frac{(F^T F)^{-1} F^T \tilde{F} \tilde{F}^T F}{T^2} \right)^{-1} \right) = \text{tr}(A_{\tilde{F}, F}^{-1}) = \text{tr}(\tilde{A}_{\tilde{F}, F}^{-1})
\]

From Lemma 4,

\[
\rho_{\tilde{F}, F} = \text{tr}(A_{\tilde{F}, F}) = \text{tr} \left( I_K + \left( \frac{F^T F}{T} \right)^{-1/2} \left( Q^T \right)^{-1} \left( \frac{\tilde{W} e e^T \tilde{W}}{T} \right)^{-1} \left( \frac{F^T F}{T} \right)^{-1/2} \right) + o_p(1).
\]

Then

\[
\text{tr} \left( \left( \frac{(F^T F)}{T} \right)^{-1} \left( \frac{\tilde{F} e e^T \tilde{F}}{T^2} \right) \right) = \text{tr} \left( I_K + \Xi \right) + o_p(1)
\]

and

\[
\rho_{\Lambda, \Lambda} \geq K - \text{tr} \left( \frac{\tilde{F} e e^T \tilde{F}}{N T^2} \right) \text{tr} \left( \left( \frac{\Lambda \Lambda F^T F}{N T} \right)^{-1} \right) \text{tr} \left( I_K + \Xi \right) + o_p(1)
\]

From the proof of Theorem 2, we have

\[
\text{tr}(I_K + \Xi) \leq K + (1 + h(m)) \sigma_e^2 \text{tr}(A^{-1}) + o_p(1)
\]

We need to provide an upper bound for \((1 + h(m)) \sigma_e^2 \text{tr}(A^{-1})\), where \(A = \tilde{U}^T (U U^T) \tilde{U}\). From the assumption in Theorem 2, the smallest eigenvalue for \(A\) is bounded away from 0. Then the largest eigenvalue for \(\text{tr}(A^{-1})\) is bounded above from infinity and \(\text{tr}(A^{-1}) = O_p(1)\).
Next, we need to provide an upper bound for $tr \left( \frac{\tilde{F}^T e e^T \tilde{F}}{NT^2} \right)$ and $tr \left( \left( \frac{\Lambda^T \Lambda F^T F}{NT} \right)^{-1} \right)$. For the term $tr \left( \frac{\tilde{F}^T e e^T \tilde{F}}{NT^2} \right)$, since

$$\frac{1}{N} e^T e \to \tilde{\Sigma}_e$$

and maximum eigenvalue of $\tilde{\Sigma}_e$ is $o(T)$ from Assumption 5, together with $\tilde{F}^T \tilde{F}/T = I_K$, we have

$$\frac{1}{NT^2} \tilde{F}_j^T e^T e \tilde{F}_j = o_p(1)$$

and then

$$tr \left( \frac{\tilde{F}^T e e^T \tilde{F}}{NT^2} \right) = \sum_{j=1}^K \frac{1}{NT^2} \tilde{F}_j^T e^T e \tilde{F}_j = o_p(1).$$

Furthermore,

$$\frac{1}{NT} \Lambda^T \Lambda F^T F \to \Sigma_{\Lambda} \Sigma_F.$$ 

From Assumptions 1 and 2, $\Sigma_F$ and $\Sigma_\Lambda$ are positive definite matrix. Then $\Sigma_F \Sigma_\Lambda$ is invertible and the maximum eigenvalue of $(\Sigma_F \Sigma_\Lambda)^{-1}$ is bounded away from $\infty$. Thus, $tr \left( \left( \frac{\Lambda^T \Lambda F^T F}{NT} \right)^{-1} \right) = O_p(1)$. We then have

$$tr \left( \frac{\tilde{F}^T e e^T \tilde{F}}{NT^2} \right) tr \left( \left( \frac{\Lambda^T \Lambda F^T F}{NT} \right)^{-1} \right) tr (I_K + \Xi) = o_p(1)$$

followed from $tr \left( \frac{\tilde{F}^T e e^T \tilde{F}}{NT^2} \right) = o_p(1)$, $tr \left( \left( \frac{\Lambda^T \Lambda F^T F}{NT} \right)^{-1} \right) = O_p(1)$ and $tr (I_K + \Xi) = O_p(1)$.

Thus, $\rho_{\tilde{\Lambda}, \Lambda} \xrightarrow{p} K$ and $\rho_{\tilde{\Lambda}, \Lambda} \xrightarrow{p} K$.

\[\square\]

**Proof of Proposition** Define $\bar{\rho}$ and $\bar{\rho}$ as

$$\bar{\rho} = tr \left( \left( F^T F/T \right)^{-1} \left( F^T \tilde{F}/T \right) \left( \tilde{F}^T \tilde{F}/T \right)^{-1} \left( F^T F/T \right) \right)$$

$$\bar{\rho} = tr \left( \left( F^T F/T \right)^{-1} \left( F^T \tilde{F}/T \right) \left( \tilde{F}^T \tilde{F}/T \right)^{-1} \left( F^T F/T \right) \right)$$

Assume there are $m_N$ nonzero elements in both $\tilde{\Lambda}$ and $\Lambda$. Since it is an one factor model, $\tilde{\Lambda} = \tilde{\lambda}_1$ and $\Lambda = \lambda_1$. Assume the indexes of nonzero values in $\tilde{\lambda}_1$ and $\lambda_1$ are $\tilde{1}_i$ and $\tilde{1}_i$ $i = 1, 2, \cdots m_N$ respectively.

From lemma 4, we have as $N, T \to \infty$,

$$\bar{\rho} = tr \left( \left( I + \left( \frac{F^T F}{T} \right)^{-1/2} \left( \bar{\Lambda}^T \Lambda \right)^{-1} \bar{\Lambda}^T \tilde{e} \bar{\Lambda} / T \left( \Lambda^T \Lambda \right)^{-1} \left( \frac{F^T F}{T} \right)^{-1/2} \right)^{-1} \right) + o_p(1)$$

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For $\frac{\hat{\Lambda}^Tee^T\hat{\Lambda}}{T}$, in a one-factor model,

$$\frac{1}{T}\hat{\Lambda}^Tee^T\hat{\Lambda} = \frac{1}{T} \sum_{i=1}^{m} \sum_{k=1}^{m} \tilde{\lambda}_{1i,1} \hat{\lambda}_{1k,1} e_i^T e_k$$

$$= \frac{1}{T} \sum_{i=1}^{m} \hat{\lambda}_{1i,1}^2 e_i^T e_i + \frac{1}{T} \sum_{i \neq k} \hat{\lambda}_{1i,1} \hat{\lambda}_{1k,1} e_i^T e_k,$$

where $e_i$ is the $i$-th row in $e$. If errors are cross-sectionally independent (weekly dependent), then for fixed $m$, $\frac{1}{T} \sum_{i \neq k} \hat{\lambda}_{1i,1} \hat{\lambda}_{1k,1} e_i^T e_k = o_p(1)$. Furthermore, since $\sum_{i \neq k} \hat{\lambda}_{1i,1} = 1$, if there is some $\sigma_e^2$, such that $\frac{1}{T} e_i^T e_i \to \sigma_e^2$, then

$$\frac{1}{T}\hat{\Lambda}^Tee^T\hat{\Lambda} = \sigma_e^2 + o_p(1)$$

Then we have

$$\hat{\rho} = \left(1 + \frac{\sigma_e^2}{\frac{1}{T}\hat{\Lambda}^T (\Lambda F^T F \hat{\Lambda}) \hat{\Lambda}} \right)^{-1} + o_p(1)$$

Similarly, for sparse loadings calculated from Lasso, with the standardization, $\bar{\Lambda}^T \bar{\Lambda} = 1$, we also have

$$\bar{\rho} = \left(1 + \frac{\sigma_e^2}{\frac{1}{T}\bar{\Lambda}^T (\Lambda F^T F \bar{\Lambda}) \bar{\Lambda}} \right)^{-1} + o_p(1)$$

From the proof of Proposition 1, we have

$$\frac{1}{T}\bar{\Lambda}^T (\Lambda F^T F \bar{\Lambda}) \bar{\Lambda} = \sigma_f^2 \sum_{i=1}^{m} \bar{\lambda}_{1i,1}^2 + o_p(1) + o_p(1)$$

On the other hand,

$$\frac{1}{T}\bar{\Lambda}^T (\Lambda F^T F \bar{\Lambda}) \bar{\Lambda} = \tilde{S} \sum_{i=1}^{m} \bar{\lambda}_{1i,1}^2, + o_p(1),$$

where $\tilde{S} = H^{-1} \frac{F^T F}{T} (H^T)^{-1}$.

From Cauchy-Schwarz inequality,

$$\left( \sum_{i=1}^{m} \bar{\lambda}_{1i,1} \hat{\lambda}_{1i,1} \right)^2 \leq \left( \sum_{i=1}^{m} \bar{\lambda}_{1i,1}^2 \right) \left( \sum_{i=1}^{m} \hat{\lambda}_{1i,1}^2 \right) = \sum_{i=1}^{m} \hat{\lambda}_{1i,1}^2.$$ 

The equality holds only if $\forall i$, $\bar{\lambda}_{1i,1} = \hat{\lambda}_{1i,1}$ is the same. However, the sparse loadings are obtained from Lasso Regression. The coefficients equal to those from Least Angle Regression [Efron]
et al., 2004; Friedman et al., 2001), which in general are not proportional to the least square coefficients. Thus, in general, $\frac{\hat{\lambda}_{1,i}}{\hat{\lambda}_{1,i}}$ is not the same $\forall i$. The above inequality is a strict inequality.

Thus,

$$\frac{1}{T} \tilde{\Lambda}^\top (\Lambda F^\top F \Lambda^\top) \tilde{\Lambda} < \frac{1}{T} \tilde{\Lambda}^\top (\Lambda F^\top F \Lambda^\top) \tilde{\Lambda} + o_p(1)$$

and therefore, as $N, T \to 0$,

$$\Delta \rho = \bar{\rho} - \hat{\rho} \geq 0$$

with probability 1.

D Empirical Application

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<td>5 Dividend/Price - divp</td>
<td>24 Gross Profitability - prof</td>
</tr>
<tr>
<td>6 Earnings/Price - ep</td>
<td>25 Return on Assets (A) - roaa</td>
</tr>
<tr>
<td>7 Gross Margins - gmargins</td>
<td>26 Return on Book Equity (A) - roea</td>
</tr>
<tr>
<td>8 Asset Growth - growth</td>
<td>27 Seasonality - season</td>
</tr>
<tr>
<td>9 Investment Growth - igrowth</td>
<td>28 Sales Growth - sgrowth</td>
</tr>
<tr>
<td>10 Industry Momentum - indmom</td>
<td>29 Share Volume - shvol</td>
</tr>
<tr>
<td>11 Industry Mom. Reversals - indmomrev</td>
<td>30 Size - size</td>
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<td>12 Industry Rel. Reversals - indrev</td>
<td>31 Sales/Price - sp</td>
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<td>13 Industry Rel. Rev. (L.V.)</td>
<td>32 Short-Term Reversals - strev</td>
</tr>
<tr>
<td>14 Investment/Assets - inv</td>
<td>33 Value-Momentum - valmom</td>
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<tr>
<td>15 Investment/Capital - invcap</td>
<td>34 Value-Momentum-Prof. - valmomprof</td>
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<td>16 Idiosyncratic Volatility - ivol</td>
<td>35 Value-Protability - valprof</td>
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<td>17 Leverage - lev</td>
<td>36 Value (A) - value</td>
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<td>18 Long Run Reversals - lrrev</td>
<td>37 Value (M) - valuem</td>
</tr>
<tr>
<td>19 Momentum (6m) - mom</td>
<td></td>
</tr>
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</table>

Table 3: List of 37 anomaly characteristics in 370 single-sorted portfolios
Figure 13: Financial single-sorted portfolios: Portfolio weights of 1st proximate factor. The sparse loading has 30 nonzero entries.

Figure 14: Financial single-sorted portfolios: Portfolio weights of 2nd proximate factor. The sparse loading has 30 nonzero entries.

Figure 15: Financial single-sorted portfolios: Portfolio weights of 3rd proximate factor. The sparse loading has 30 nonzero entries.
Figure 16: Financial single-sorted portfolios: Portfolio weights of 5th proximate factor. The sparse loading has 30 nonzero entries.